Chapter 16B

Economic Growth - I

Learning Objectives :

After learning this chapter you will understand :

- **Solow Growth Model.**
- **The Steady State Level of Capital.**
- **Effects of Changes in Saving Rate.**
- **Golden Rule Level of Capital.**
- **Transition to the Golden Rule Level of Capital.**
- **Effects of Population Growth in the Solow Model.**

Basic Concepts

1. **Solow Growth Model :** The Solow growth model is named after economist Robert Solow. Solow Growth Model explains that why the economics of some countries gross faster than other countries. It shows how saving, population growth and technological progress effect the level of an economies output and its growth overtime. Initially we assume that labour technology are fixed and we show that how growth in capital stock affects nation's output.
2. **Supply and Demand of Goods :** In the static (Constant) Model, demand and supply were considered to be fixed and they played a key role in understanding that model. Similarly, in Solow growth model we can determine how much output is produced and how it is allocated between various factors using supply and demand of goods.

   (i) **Supply of Goods :** In the Solow Model, the supply of goods is based on production function. The production function states that output is based upon capital stock and labour force :

   $$Y = f(K, L)$$

   Solow model assumes that production function exhibits constant returns to scale, *i.e.*, if labour and capital both are multiplied by a constant 'λ' then output also gets multiplied by the same constant 'λ' because production function gives constant returns to scale, *i.e.*,

   $$Y = f(K, L)$$

   So, $f(\lambda K, \lambda L) = \lambda f(K, L)$

   $= \lambda . y$

   Since labour is assumed to be constant so we can consider output and capital per unit of labour.

   Let $\lambda = 1/L$

   $f(K/L, 1) = Y/L$

   Let $K/L = k$ & $Y/L = y$

   So, the production function becomes $f(k, 1) = y$

   or $y = f(k)$.

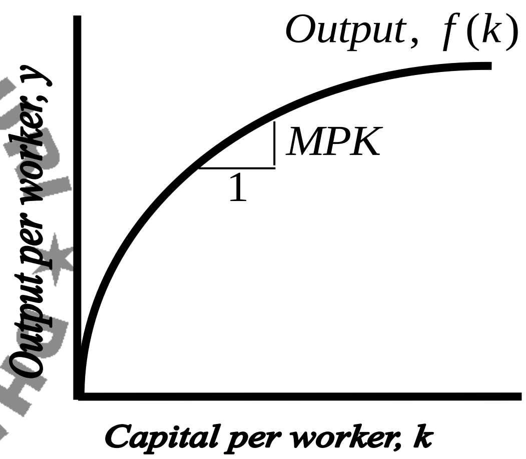


Figure 1 : The Production Function

   Because we have assumed that production function gives constant returns to scale so the size of economy, as measured by number of workers, does not effect the relationship between output per worker and capital per worker.

   Slope of Production Function : The slope of production function shows how much extra output a worker produces when given an extra unit of capital. So the slope of production function is marginal product capital, *i.e.*, if k increases by 1 unit then output increases by MP_k units.

Slope = $MP_K = f(k+1) - f(k) = \partial f/\partial k$

The production function becomes flatter as k increases indicating that it gives diminishing marginal product of capital. When k is low so the average worker has little capital to work with so an additional unit of capital is very useful and produces a lot of additional output. But when capital is very high then average worker has a lot of capital so additional unit produces very less output.

(ii) **Demand for Goods :** In the Solow Model, the demand for goods comes from the consumption & investment, *i.e.*, the output per worker is the summation of consumption per worker & investment per worker.

$$y = c + i$$

where c = C/L = consumption per worker

i = I/L = investment per worker

Saving : In the Solow Growth Model, we assume that people save a fraction of their income and consume a fraction of their income every year. Let 's' represent the fraction of saving to the fraction of consumption is $1 - s$.

Savings = $s.y$

Consumption = $(1 - s)y$

Now,

$y = c + i$

$\Rightarrow \quad y = (1 - s)y + i$

$\Rightarrow \quad y = y - s.y + i$

$\Rightarrow \quad i = s.y = s.f(k) \qquad [\because y = f(k)]$

$\Rightarrow$ investment per worker = saving per worker

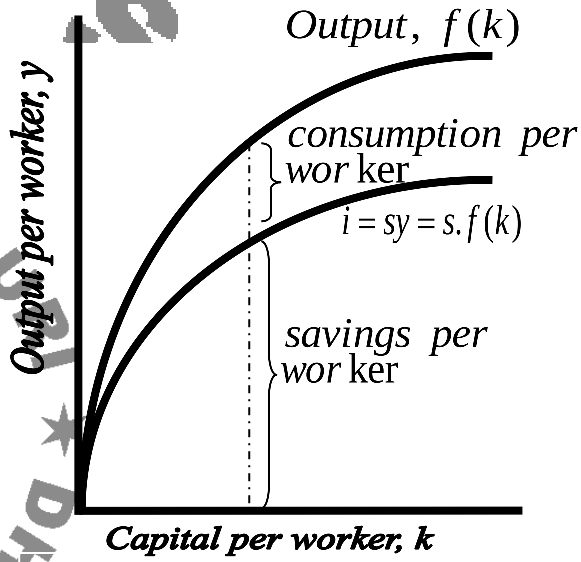


Figure 2 : Consumption and Savings

This equation shows that investment equals saving. Thus, the rate of saving 's' is also fraction of output devoted to investment.

Also, $c = y - i \quad = f(k) - s.f(k)$

Figure 2, illustrates how for any value of k, the amount of output is determined by the production function $f(k)$ and the allocation of that output between consumption and saving is determined by the saving rate s.

3. **Growth in the Capital Stock :** The output of any economy depends upon its capital stock, those economies which have large amount of capital will have proportionately greater output in comparison to those which have smaller capitals. The capital stock of an economy also tends to change over time. There are two forces which influence the capital stock.

(i) Investment, and (ii) Depreciation

(i) **Investment :** Investment refers to purchase of capital goods and causes the capital stock to rise. *i* represent investment per worker and we know that

$y = c + i$

$\Rightarrow \quad y = (1 - s)y + i$

$\Rightarrow \quad y = y - s.y + i$

$\Rightarrow \quad i = s.y = s.f(k) \qquad [\because y = f(k)]$

$\Rightarrow$ investment per worker = saving per worker

So, investment increases proportionately with the increase in capital stock, *i.e.*, for given rate of saving, those economies which have higher level of capital will have higher level of investment.

(ii) **Depreciation :** Depreciation represent the loss in the value of asset due to wear and tear or passage of time. A constant fraction δ represent rate of depreciation. So, the amount of capital that depreciates every year = δk. Also, a constant fraction δ of the capital stock wears out every year. Therefore, depreciation is proportional to the level of capital. If capital lasts an average of 20 years, then the depreciation rate is 5%, i.e., $\delta = 0.05$.

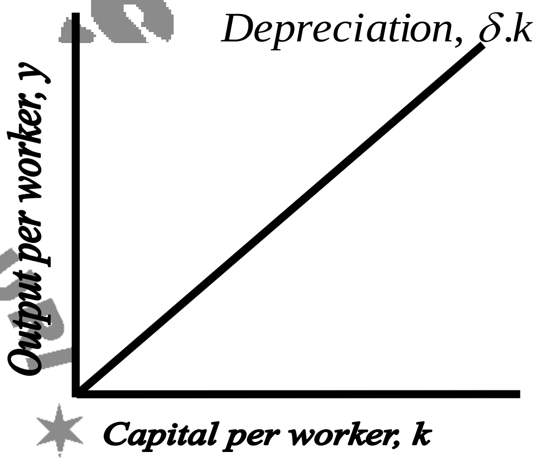


Figure 3 : Depreciation

4. **Effect of Investment and Depreciation on Capital Stock :** Investment causes an increase in capital stock whereas depreciation reduces it. We can express the impact of investment and depreciation on the capital stock with the following equation :

Change in Capital stock = Investment – Depreciation

$$\Delta k = i - \delta.k$$

Since investment *i* equals *s.f*(*k*), so we can write the above equation as :

$$\Delta k = s.f(k) - \delta.k$$

Here, Δk is the change in the capital stock between one year and next year.

5. **The Steady State Level of Capital :** The steady state level of capital is that level at which the amount of investment equals the amount of depreciation. At this level the capital stock will not change because the two forces which change the capital, *i.e.*, depreciation and investment are equal to each other. The steady state level of capital is also known as long run equilibrium. ***Figure 4***, illustrates the steady state level of capital. Here, *k** is the steady state level of capital, *y** is the steady state level of output per worker, *c** is the steady state level of consumption per worker and s* is the steady state level of saving per worker.

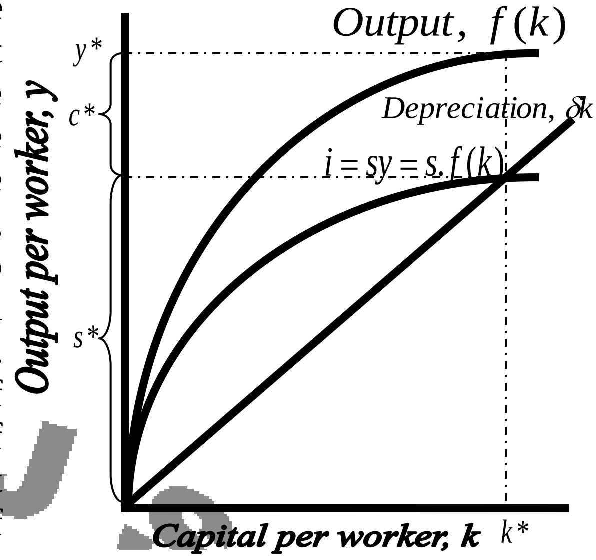


Figure 4 : Steady State Level of Capital.

The steady state level of capital *k** is the level at which investment equals depreciation, indicating that the amount of capital will not change over time, i.e., $\Delta k = 0$. Below *k** investment exceeds depreciation, so the capital stock grows and above *k** depreciation exceeds investment so the capital stock shrinks. The steady state level of capital is important for Two reasons :

(i) The economy which is at steady state level will remain at steady state.

(ii) The economy which is not at steady state level will move towards steady state in the long run so steady state level is also known as long run equilibrium of the economy.

6. **Effects of Change in Saving Rate :** When the saving rate in an economy increases the steady state level of capital will also rise. For example, if S_1 is the initial saving rate *k* * is the initial steady state level of capital. With the increase in saving rate from S_1 to S_2 the investment will rise immediately but since the depreciation rate is unchanged so the investment exceeds depreciation and capital stock will rise. The capital stock will continue to rise until new steady state k_1^* is obtained which has the capital stock and higher level of output than the original steady state.

The Solow model shows that the saving rate is a key determinant of the steady state capital. If the saving rate is high, the economy will have a large capital stock and a high level of output in the steady state. Whereas If the saving rate is low, the economy will have a small capital stock and a low level of output in the steady state.

7. **Relation Between Saving Rate and Economic Growth :** According to Solow model, higher savings leads to faster growth but only temporarily. An increase in the rate of saving raises growth only until the economy reaches the new steady state. The higher rate of savings yield higher capital stock and higher output levels at the steady state level but it will not maintain a higher rate of growth forever. A

higher saving rate is said to have a *level effect*, because only the level of income per person is influenced by the saving rate in the steady state.

8. **Relation between Saving and Investment :** The data collected from various economies shows that there is direct relationship between the fraction of output devoted to investment and the level of income per person, *i.e.*, the economies with higher saving rate have high levels of income whereas the economies with low saving rate will have low level of income. But the relationship between these two variables are far from perfect. There are many economies in the world which have same level of saving but different level of income. There are various reasons for this like political situation, cultural differences, stability of financial market etc.

9. **Golden Rule Level of Capital :** The Golden Rule level of capital represents the level that maximizes consumption in the steady state and it is denoted by K^*_{gold}. The higher rate of saving yields greater income but the objective of the economist is to maximize the economic well-being. For the sake of simplicity we assume that the policy maker can set the saving rate at any level. When choosing a steady state, the policymaker's goal is to maximize the well-being of the individuals who make up the society. Individuals themselves do not care about the amount of capital in the economy or even the amount of output. They care about the amount of goods and services they can consume. Thus, a benevolent policymaker would want to choose the steady state with the highest level of consumption. Such Steady state level of capital is called Golden Rule level of Capital.

   In the Solow Growth Model, a steady state saving rate of 100% implies that all the income is going to investment implying a steady state consumption level of 0%. But if the saving rate is 0%, *i.e.*, no investment is being made so the capital stock depreciate without replacement, this makes steady stage unsustainable except at zero output which again implies consumption level of zero. Thus, somewhere in between is the level of savings where the level of consumption is at its maximum possible value, the corresponding steady state level of capital is called Golden Rule level of Capital and it is denoted by k^*_{gold}.

10. **Mathematical Derivation of Golden Rule Level of Capital :**

We know that,

$c = y - i$

$\Rightarrow \quad c = f(k) - s.f(k)$

At steady state level, k^*

i = Depreciation, *i.e.*, δk^*

$\Rightarrow \quad s.f(k^*) = \delta k^*$

$\therefore \quad c^* = f(k^*) - \delta k^*$

Differentiating both sides w.r.t k^*

$\partial c^*/\partial k^* \quad = \quad \partial f(k^*)/\partial k^* - \delta \quad = \quad \text{MPk} - \delta$

First order condition, for maximum consumption is

$\partial c^*/\partial k^* = 0$

$\Rightarrow$ MPk – $\delta = 0$

$\Rightarrow$ MPk = δ

So, the level of capital for which we have MPk = δ, represents the golden rule level of capital.

11. **Determining Golden Rule Level of Capital using MPK and δ :** The slope of the production function is marginal product of capital (MPk) and the slope of depreciation is δ.
   (i) If MPk > δ, then the production function is steeper than the depreciation line. Therefore, increase in capital increases output more than depreciation, so consumption increases.
   (ii) If MPK < δ, then the production function is flatter than the depreciation line. Therefore, increase in capital increases output less than depreciation, so consumption decreases.
   (iii) If MPk = δ then the production function is parallel to the depreciation line. Therefore, increase in capital increases output which is equal to depreciation so the consumption does not change. Thus this is the golden rule level of capital.

12. **How to Determine if the Economy is at Golden Rule Level of Capital :** The following steps are used to determine if the economy is at Golden Rule Level of capital.
   (i) We know that income is the sum of consumption and investment so income per worker is the sum of consumption per worker and investment per worker, *i.e.*, $y = c + i$ $\Rightarrow$ $c = y - i$, *i.e.*, consumption per worker is simply output per worker minus income per worker.
   (ii) At steady level of capital k^*, investment is equal to depreciation. So $i = \delta k^*$ and steady state output is $f(k^*)$. So replacing the value of output and investment we get

   $$c^* = f(k^*) - \delta k^*$$

   Thus, steady state consumption is steady state output minus state depreciation. This equation implies that there are two opposing effects of increase in steady state capital on steady state consumption :
      (a) An increase in steady state capital increases output.
      (b) An increase in steady state capital means more depreciation.
   (iii) The higher level of capital effects both output and depreciation. If the capital stock is below the golden rule level then an increase in capital stock raises the output more than the depreciation so the consumption increases. Whereas when capital stock is above the golden rule level then increase in capital increases production but increase in output is less than depreciation

so consumptions falls. Thus, there is a point between these two level where increase in output is equal to depreciation, *i.e.*, Slope of Production function is equal to slope of depreciation line (MPK = δ). So, at this point consumption is maximum.

Note **There is only one saving rate which produces golden rule level of capital k*gold. Any change in the saving rate would shift the investment curve s.f (k) and would move the economy to a steady state with a lower level of consumption**

Note **The economy does not automatically gravitate toward the Golden Rule steady state. If we want any particular steady state capital stock such as Golden Rule, we need a particular saving rate to support it.**

13. **Transition to the Golden Rule level of Capital :** We have assumed that the policy maker can simply choose the economy steady state and can easily go there so the policy maker would choose such saving rate where the level of consumption is highest, *i.e.*, golden rule steady state. If economy is already at the golden rule steady state level then there are no issues for the policy makers but if the economy has reached a steady state level other than the golden rule level then there are two possibilities.
    (i) Starting with too much capital, *i.e.*, capital is more than the golden rule.
    (ii) Starting with two little capital, *i.e.*, capital is less than the golden rule.

    Case I : Starting with too much capital : If the economy has capital which is more than the Golden rule Steady than the policy maker should aim at reducing the saving rate such that capital is reduced and becomes equal to golden rule level. If the policy maker reduces the saving rate at time t_0 then there will be a sudden increase in consumption and decrease in investment. Because investment and depreciation were equal in the initial steady state, So depreciation will be more than investment and now the economy will not be in steady state. Gradually, It will cause a reduction in capital stock and consequently in output, consumption and investment. These variables will keep on falling until the economy reaches a new steady state level which we have assumed is the golden rule level so here the consumption should be more than the original consumption.

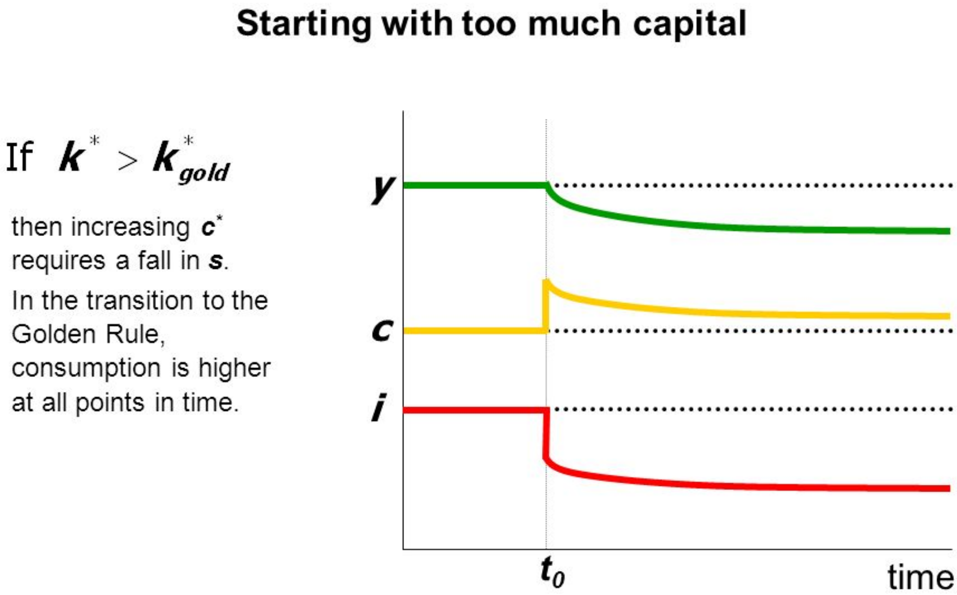


Figure 5 : Transition to Golden Rule Level of Capital

Note **Compared to the old steady state, consumption is higher not only in the new steady state but also along the entire path. So when capital is above the golden rule level, reduction is saving is a good economic policy.**

Case II : Starting with too little capital : If the economy has less capital then to reach the golden rule level of capital, the policymaker should increase the saving rate. When policy maker increases the saving rate the capital will start rising which will increase the output per worker. The consumption will fall initially but gradually it will rise because output is rising and finally consumption reaches a level which is more than the initial level of consumption. Suppose at time t_0 the policy maker increases the saving rate so the consumption will fall immediately. The increase in saving rate causes increase in capital stock over time which causes increase in output and hence increase in consumption. So, eventually economy reaches the golden rule level of capital where the consumption is maximum.

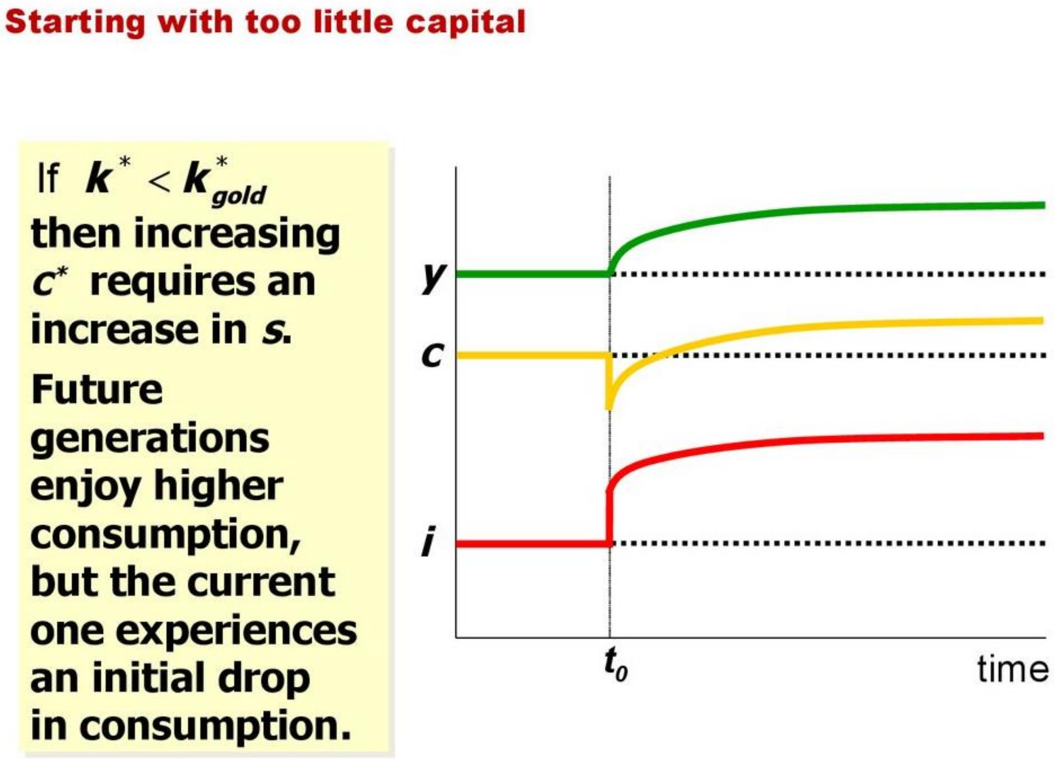


Figure 6 : Transition to Golden Rule Level of Capital

14. **Population Growth in the Solow Growth Model :** Instead of assuming that the population is fixed, we now suppose that the population and the labour force grow at a constant rate *n*. Capital per worker is k = K/L and output per worker is y = Y/L. If labour force (L) increases then it with decrease capital per worker and also output per worker so that increase in labor force causes a decreases in capital. The change in capital per worker can be represented as $\Delta k = i - (\delta + n)k$. Here '$n$' represent the rate at which labor force is growing. This equation shows that investment, depreciation and population growth changes the capital per worker, investment increases the capital per worker whereas depreciation and population growth decreases capital per worker.

    Depreciation causes decrease in capital due to wear and tear. Whereas increase in labour force decreases capital per worker by spreading it more thinly among the population.

15. **Break Even Level of Investment :** The level of investment which is necessary to keep the capital per worker constant is called break even level of investment. Depreciation and population growth are two reasons due to which the capital per worker shrinks. If n is the rate of population growth and δ is the rate of depreciation then $(\delta + n)k$ represents the break even level of investment, *i.e.*, the amount of investment necessary to keep the capital per worker k as constant.

16. **Steady State with Population Growth :** An economy is in steady state if the capital per worker is not changing. Let k^* denotes the steady state level of capital so at k^* investment is equal to depreciation and population growth, *i.e.*, $i = (\delta + n)k$

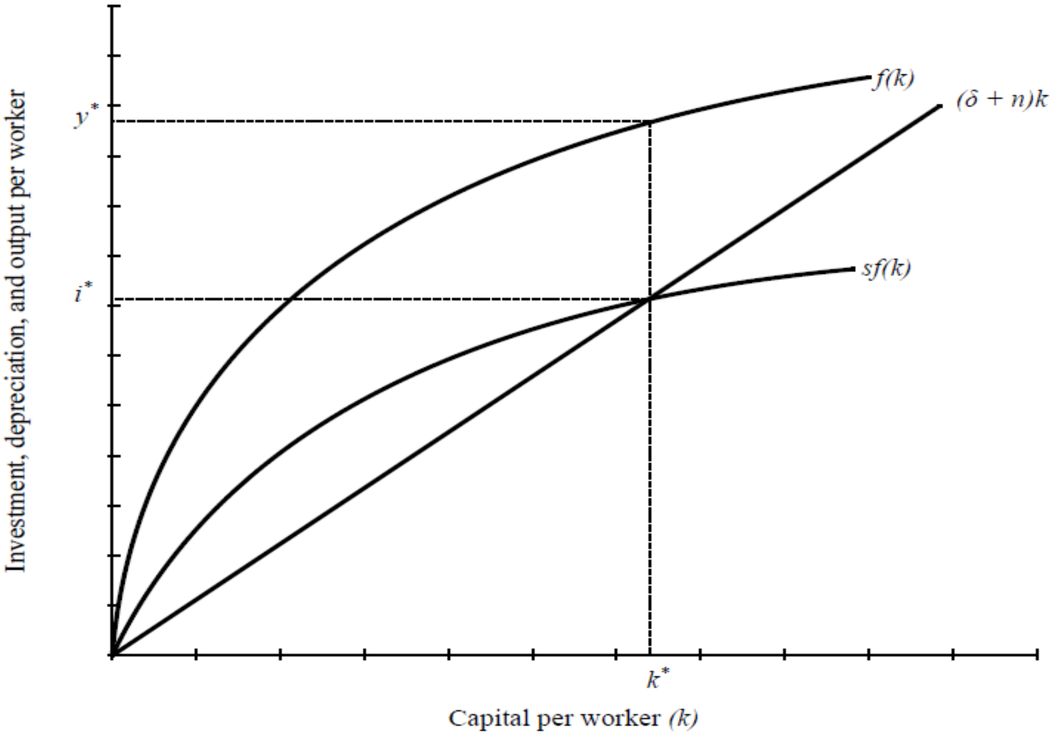


Figure 7 : Steady State Level with Depreciation and Population Growth

As shown in ***Figure 7***, in order to retain an unchanging level of capital per worker k over time, we have to invest enough to create new capital to offset this loss over time. Thus, to maintain a "steady state" where capital per worker is constant over time, we must have that :

$$\Delta k = s.f(k) - (\delta + n)k = 0$$

$$\Rightarrow \quad s.f(k^*) = (\delta + n)k^*$$

where * indicates steady state values. Note how this shows that, as our capital per worker k gets larger, larger amounts of investment are required to maintain $\Delta k = 0$. The economy will always work itself to a steady state point. If the rate of capital replenishment is greater than the loss due to depreciation and population growth, *i.e.*, $s.f(k) > (\delta + n)k$, then the capital stock will grow. If the rate of replenishment is lower than depreciation plus population growth, *i.e.*, $s.f(k) < (\delta + n)k$, then the capital stock will shrink. Only when the two are equal will there be no further adjustment to the capital stock in the economy.

Deriving Steady State Consumption

We know that

$$y = c + i$$

$$\Rightarrow \quad c = y - i$$

$$\Rightarrow \quad c = f(k) - i$$

at steady state level of capital, c^* is consumption, $f(k^*)$ is output, and $i = (\delta + n)k^*$ is break even investment.

$$\therefore \quad c^* = f(k^*) - (\delta + n)k^*$$

Note **There are an infinite number of possible steady states, some higher than others. Which steady state our economy is in (and therefore what output we have) depends on where the *s.f*(*y*) curve meets the (δ + *n*)*k* curve, which in turn depends on the savings rate *s* in the economy.**

17. **Effects of Population Growth :** The population growth has the following effects in the Solow Model

(i) Population growth explains sustained growth in total output. At the steady state level of capital the population growth, the capital per worker and the output per worker all are constant. Since number of workers are growing at constant rate n, however, total capital and total output must also be growing at rate *n*.

(ii) If the population growth rate increases then steady state level declines. Thus, the Solow model predicts that economies with higher rates of population growth will have lower levels of capital per worker and therefore lower incomes.

(iii) Golden rule level of Capital is obtained where marginal productivity of capital is equal to the sum of depreciation rate and population growth rate, *i.e.*, MPk = $\delta + n$.

Note **A change in population growth rate, like a change in the saving rate, has a <u>level effect</u> on the income per person, but does not affect the steady state growth rate of income per person.**

Exercise 1

Theory Questions

Steady State

Q1. In the Solow model, how does the saving rate affect the steady state level of income?

Q2. In the Solow model, how does the saving rate affect the steady state rate of growth?

Q3. Consider the following statement : "Devoting a large share of national output to investment would help to restore rapid productivity growth and increasing living standards." Do you agree with this statement in the context of the Solow Model. Explain?

Q4. Explain why an increase in saving in the Solow model has a level effect but not a growth effect.

Golden Rule Level of Capital

Q5. In the Solow model with no population growth and no technological progress, explain the concept of Golden Rule level of consumption. Use a diagram to discuss your answer.

Q6. Define the Golden rule level of capital $\left(K^*_{gold}\right)$.

Q7. What is meant by Golden Rule Steady State? How does the economy get there if it has smaller amount of initial capital?

Q8. Explain short run and long run implications of increasing capital per worker to reach golden level.

Q9. Suppose that in the Solow model without technological progress, an economy begins with less capital than in the Golden Rule Steady State. The saving rate at time t_0 rises to the golden Rule level. Show the time path of adjustment of output per worker, consumption per worker and investment per worker.

Q10. Suppose that the economy has reached a steady state with more capital than it would have in the golden rule steady state, what is the transition to the golden rule steady state.

Q11. What is the Golden Rule level of capital? Explain the policy interventions required to achieve the Golden Rule level if the actual capital stock differs from the Golden Rule level.

Q12. What do you understand by golden rule steady state ? Derive the condition for golden rule steady state.

Q13. An economy is at a steady state with less capital than the golden rule steady state. Show the transition paths of output, consumption and investment towards the golden rule steady state.

Population Growth

Q14. Discuss the steady state condition in the Solow model with population growth and no technological progress.

Q15. In the Solow model, how does the rate of population growth affect the steady state level of income?

Q16. Show that in the steady state of the Solow model without technological progress, the capital per worker and output per worker are constant but total capital and total output grow at the rate of population growth.

Q17. Assuming zero growth rate of technology show the effect of an increase in the stock of labour and decrease in investment rate on the economy's level of output in the context of the Solow model.

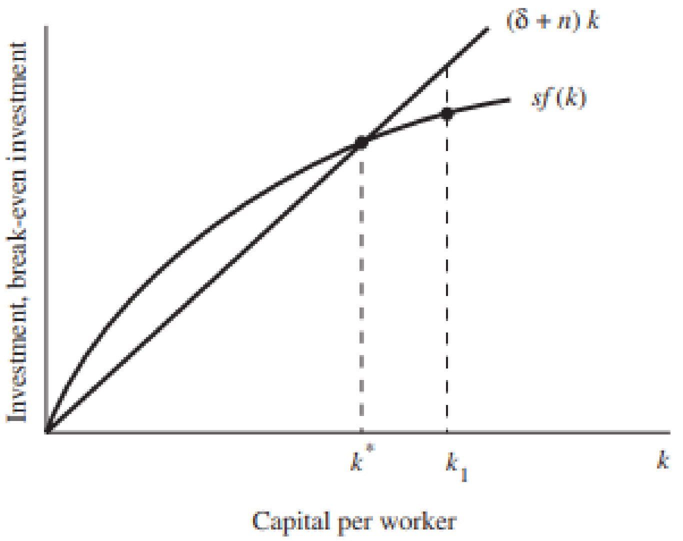
Q18. Consider an economy characterized by the Solow Model this is initially in steady state. A war reduces the capital stock in this economy while leaving the labour unchanged. The war is accompanied by a decline in the saving rate of the economy. Illustrate the immediate and the long term impact of the war and the change in the saving rate on the capital per capita in the economy.

Q19. Consider an economy with a fixed saving rate and no technological progress. Suppose there is a war that does not directly affect the capital stock but the casualties reduce the labour force. What is the immediate impact on total output and on output per worker? Assuming that the saving rate is unchanged and that the economy was in a steady state before the war, what happens subsequently to output per worker in a post war economy? Does the growth rate of output per worker increase or decrease after the war?

Answer

(i) The production function in the Solow growth model is Y = F(K, L), or expressed terms of output per worker, $y = f(k)$. If a war reduces the labor force through casualties, then L falls but $k = K/L$ rises. The production function tells us that total output falls because there are fewer workers. Output per worker increases, however, since each worker has more capital.

(ii) The reduction in the labor force means that the capital stock per worker is higher after the war. Therefore, if the economy were in a steady state prior to the war, then after the war the economy has a capital stock that is higher than the steady state level. This is shown in adjoining Figure, as an increase in capital per worker from k^* to k_1. As the economy returns to the steady state, the capital stock per worker falls from k_1 back to k^*, so output per worker also falls.



Numerical Problems

Q1. Consider an economy described by the following production function :

$$Y = f(K, L) = K^{1/2}L^{1/2}$$

(i) Assume that the rate of depreciation is 10% per year and rate of saving is 30%. Find the steady state capital stock per worker of Solow's model.

(ii) Does your answer to (i) also provide the golden rule steady state solution?

(iii) What is the steady state level of investment?

[**Ans. :** (i) $y = 3$, $k^* = 9$, (ii) $k^*_{gold} = 25$, (iii) $i = 0.9$]

Q2. Consider an economy described by the following production function :

$$Y = f(K, L) = K^{2/3}L^{1/3}$$

(i) Assume that the rate of depreciation is 10% per year and rate of saving is 30%. Find the steady state capital stock per worker of Solow's model.

(ii) Does your answer to (i) also provide the golden rule steady state solution?

(iii) What is the steady state level of investment?

[**Ans. :** (i) $y = 9$, $k^* = 27$, (ii) $k^*_{gold} = 296.296$, (iii) $i = 2.7$]

Q3. Consider an economy described by the following production function :

$$Y = f(K, L) = K^{1/4}L^{3/4}$$

(i) Assume that the rate of depreciation is 20% per year and rate of saving is 25%. Find the steady state capital stock per worker of Solow's model.

(ii) Does your answer to (i) also provide the golden rule steady state solution?

(iii) What is the steady state level of investment?

[**Ans. :** (i) $y = (1.25)^{1/3}$, $k^* = (1.25)^{4/3}$, (ii) $k^*_{gold} = (1.25)^{4/3}$,

(iii) $i = \frac{(1.25)^{4/3}}{5}$]

Q4. In the Solow model, without technical change but a constant rate of growth of population, assume an economy in steady state at time t_0. Let the economy attain the golden rule level of consumption per head at t_1. At t_0, let c_0, k_0 and y_0 be the levels of consumption, capital stock and output per head respectively.

(i) Derive and explain the condition which characterizes the golden rule.

(ii) If k_0 is greater than the capital stock per head at golden rule, show that the economy is dynamically inefficient by reducing the rate of savings. Consumption per head can be increased at all points of time : in the open time interval (t_0, t_1) and in the semi open time interval (t_1, ∞).

(iii) If k_0 is less than the capital stock per head at golden rule, increasing the rate of savings to reach the golden rule, involves an intergenerational tradeoff, Q5. Consider an economy with the production function $Y = K^{\alpha} L^{1-\alpha}$. Assume that α = 1/3.

(i) Is this production function characterized by Constant Returns to Scale?

(ii) In terms of saving rate s and depreciation rate δ, derive expressions for capital per worker, output per worker and consumption per worker in the steady state.

(iii) Suppose $\delta = 0.08$, and $s = 0.32$ what is the steady state capital per worker, output per worker and consumption per worker?

[**Ans : (i)** Yes, **(ii)** $k^* = \left[\frac{s}{\delta}\right]^{3/2}, y^* = \left[\frac{s}{\delta}\right]^{1/2}$ *and* $c^* = (1-s)\left[\frac{s}{\delta}\right]^{1/2}$,

(iii) $k^* = 8$, $y^* = 2$, $c^* = 1.36$]

Q6. An economy has the production function $Y = 0.5\sqrt{K}\sqrt{L}$.

(i) What is the per worker production function?

(ii) In terms of the saving rate 's' and the depreciation rate 'δ' derive steady state levels of capital per worker, output per worker and consumption per worker.

(iii) Suppose that δ = 5%. What is the steady state output per worker and consumption per worker when s = 10%.

(iv) What is the Golden Rule steady state level capital stock per worker when δ = 5%?

[**Ans : (i)** $y = 0.5\sqrt{k}$, (ii) $k^* = \frac{s^2}{4\delta^2}, y^* = \frac{s}{4\delta}, c^* = (1-s)\frac{s}{4\delta}$,

(iii) $k^* = 1$, $y^* = 1/2$, $c^* = 0.45$, (iv) $k^*_{gold} = 25$]