Chapter 4

Functions

Learning Objectives :

After learning this chapter you will understand :

- **Functions.**
- **Types of Functions.**
- **Domain and Range of a Function.**
- **Operations on Functions.**
- **Composition of Functions.**
- **Exponential and Logarithmic Functions.**
- **Inverse of a Function.**
- **Trigonometric Functions.**

Basic Concepts

1. **Function :** A function f from a non empty set X to a set Y is a rule which associates to each element x in X a unique element in Y. The unique element of Y which f associates with x in X is denoted by $f(x)$. The mapping f of X to Y is denoted by : $f : X \to Y$.

   So we cannot have a function which gives two different outputs for the same argument.

2. **Domain and Range of a Function :** The set X is called the domain of f. The element y, which corresponds to the element x, is called the value of the function at x. It is denoted by $f(x)$, read as 'f of x'. The set $Y = \{ y : y = f(x) \text{ for some } x \in X\}$, is called the co-domain or range of f. The element x belonging to the domain set is called *Independent variable* and the element y which represents the element corresponding to x is called the *dependent variable*.

> *Note :* $f : X \to Y$ denotes that f is a function from X to Y.

> *Note :* Range of f is a subset of Y, which may or may not be equal to Y.

3. **Function as Sets of Ordered Pairs :** If X and Y are two non empty sets, then a function f from X to Y is a subset of $X \times Y$ satisfying the following two conditions :

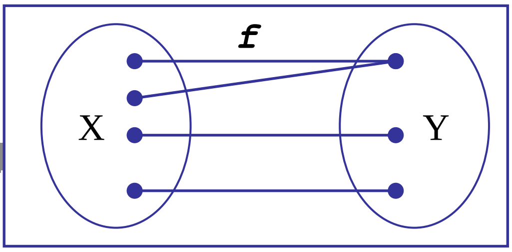
   (i) For each $x \in X$, $(x, y) \in f$ for some $y \in Y$,

   (ii) $(x, y) \in f$ and $(x, y') \in f \Rightarrow y = y'$.

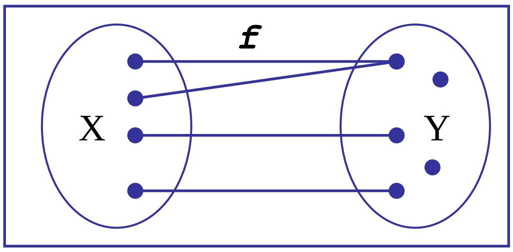
> *Note :* Every function is a relation but every relation is not necessarily a function.

4. **Types of Functions :** The functions are of the following types :

   (i) **Onto Functions :** If the function $f : X \to Y$ is such that each element in Y is the image of at least one element in X, then we say that f is a function of 'X onto Y'. In this case the range of f is equal to the co-domain of f. An onto function is also called a '**Surjection**' or a '**Surjective Function**'.

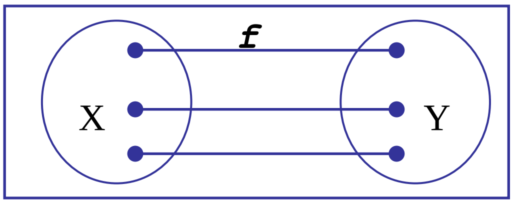


   (ii) **Into Functions :** If the function $f : X \to Y$ is such that there is at least one element in Y which is not the image any element in X, then we say that the function f is into. In this case the range

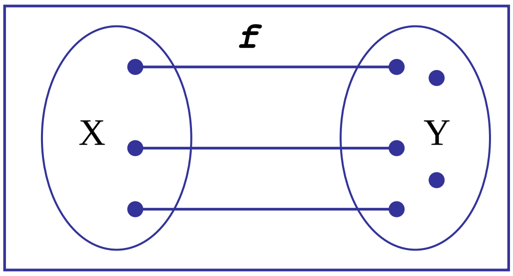


of f is the proper subset of the co-domain of f.

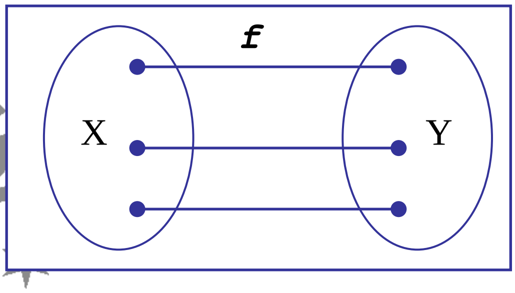
(iii) **One-one Functions :** If the function $f : X \rightarrow Y$ is such that, each element in X has distinct image in Y, then we say that the function f is one to one or one-one function.



(iv) **One-one Into Functions :** If the function $f : X \rightarrow Y$ is such that, each element in X has distinct image in Y and there is at least one element in Y which is not the image any element in X, then we say that the function f is one-one and into. An one to one into function is also called an '**Injective Function**' or an '**Injection**'.



(v) **One-one Onto Functions :** If the function $f : X \rightarrow Y$ is such that, each element in X has distinct image in Y and there no element in Y which is not the image any element in X, then we say that the function f is one-one and onto. An one to one onto function is also called a '**Bijective Function**' or an '**Bijection**'.

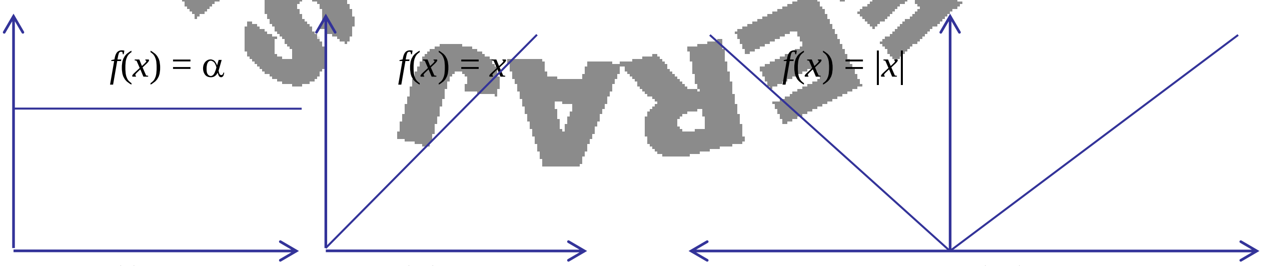


> *Note :* To prove that the given function $y = f(x)$ is a one-one function put $f(x_1) = f(x_2)$ and verify that $x_1 = x_2$.

5. **Real Functions :** If the domain and co-domain of a function are sets of all real numbers, then the function is called real valued function or real function.

6. **Types of Real Functions :**

(i) **Constant Function :** The function which associates to each real number x, a constant number α, is called a constant function. Symbolically, $f(x) = \alpha$. Figure (i).



(i) (ii) (iii)

(ii) **Identity Function :** The function which associates to each real number x the same number x, is known as identity function. Symbolically, $f(x) = x$. Figure (ii).

(iii) **Modulus Function :** The function which is defined as

$$f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$$ Figure (iii)

(iv) **Algebraic Function :** A function defined in terms of polynomials and roots of polynomials is called an algebraic function.

(v) **Polynomial Function :** The function which is of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ is called a polynomial function of degree n. Where $a_0, a_1, a_2 ..... a_n$ are real numbers and $a_n \neq 0$ and n is a whole number.

(vi) **Rational Function :** The function which is of the form $f(x) = \frac{P(x)}{Q(x)}$ is rational function, where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

(vii) **Linear Function :** Any function of the form $f(x) = mx + b$, where m and b are fixed real numbers, is called a linear function.

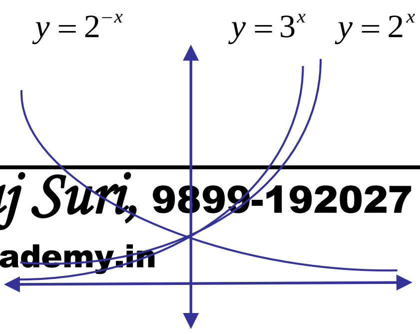
(viii) **Quadratic Function :** Any function of the form $f(x) = ax^2 + bx + c,\ a \neq 0$, where a, b and c are real numbers is called a quadratic function. The graph of the quadratic function represents a U shaped parabola. $x = -\frac{b}{2a}$ represents the vertex of the parabola. Moreover, this point is either the maximum point or the minimum point of the parabola. If a > 0, then $f(x) = ax^2 + bx + c$ has a minimum point at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ and if a < 0, then $f(x) = ax^2 + bx + c$ has a maximum point at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

(ix) **Power Function :** Any function of the form $f(x) = x^n$, where n is a positive integer, is called a power function.

(x) **Reciprocal Function :** The function which associates each non zero x to its reciprocal $\frac{1}{x}$ is called reciprocal function. Symbolically, $f(x) = \frac{1}{x}$

(xi) **Exponential Function :** The function which associates each real number x to a^x or e^x is called exponential function. Symbolically, $f(x) = a^x \text{ or } e^x$

An exponential function with base 'a' (> 1), is defined as $y = a^x$ where $-\infty < x < \infty$. Since the independent variable appears as an



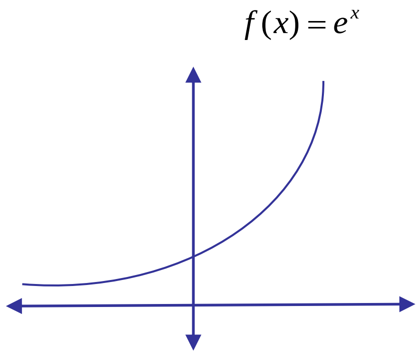
exponent, $y = a^x$ is termed as an exponential function. When a < 0, the function may not have a real value for certain values of x. When a = 0 or a = 1, we have a case of constant function. Thus, in general we always define $a > 1$.

Features of Exponential Function : The important features of an exponential function are as under :

(i) For domain $(-\infty, \infty)$, the range of the function is $(0, \infty)$, i.e., the value of $f(x)$ is positive for all values of x.

(ii) The graph of an exponential function passes through the point (0, 1).

(iii) It is a monotonic function of x, monotonically increasing when $a > 1$ and monotonically decreasing when $0 < a < 1$. The monotonicity of the function implies that there is a unique value of y for a given value of x and vice versa.



Natural Exponential Function : Geometrically, $f'(0)$ may be interpreted as the slope of the tangent to the graph of $y = a^x$ at (0, 1). The slopes of 2^x and 3^x are respectively 0.7 and 1.1 which implies that if $f(x) = 2^x \Rightarrow f'(0) = 0.7$ and if $f(x) = 3^x \Rightarrow f'(0) = 1.1$. Thus, it is reasonable to assume that as a increases from 2 to 3, so $f'(0)$ increases from 0.7 to 1.1 without skipping any intermediate values. For some values between 2 to 3, we ought then to have $f'(0) = 1$ in particular. This value of a is a fundamental constant in mathematical analysis. It is an irrational number so distinguished that it is usually denoted by the single letter 'e' and is given by e = 2.718281828459045……

Because $a = e$ is precisely the choice of a that gives $f'(0) = 1$, we obtain the ***natural exponential function*** $f(x) = e^x \Rightarrow f'(x) = e^x$.

Now, $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x$. Because $e^x > 0$ for all x, both $f'(x)$ and $f''(x)$ are positive. Hence, both f and f' are strictly increasing.

(xii) **Logarithms :** The logarithm to a given base is the index or the power to which the base must be raised to obtain that number. If $a^x = b$, a and b are positive real numbers such that $a \neq 1$, then x is said to be the logarithm of the number n to the base 'a'. Symbolically,

If $a^x = b$ [Exponential Form]

then $\log_a b = x$ [Logarithmic Form]

Note **The number '*b*' in the statement $\log_a b = x$ is called the antilogarithm of *x* to the base *a* and is written as *b* = antilog *x*.**

Types of Logarithms : The logarithms are of the following types :

(i) **Natural Logarithms :** Logarithms of numbers to the base '*e*' are called natural logarithms, these are represented as 'ln'.

(ii) **Common Logarithms :** Logarithms of numbers to the base '10' are called common logarithms.

Note

(i) The base of a logarithm is never taken as '0' because 0 raised to any power is meaningless.

(ii) The base of a logarithm cannot be negative number, otherwise certain values will become imaginary.

(iii) The base of a logarithm is never taken as 1 because 1 raised to any power is one only.

> *Note :* (i) Logarithms to the base 10 are called common logarithms.
> (ii) Logarithms to the base e are called natural logarithms.

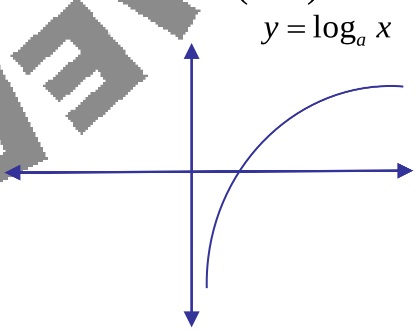
Rules of Logarithms : If *x* and *y* are two positive real numbers, then

(a) $\log_a xy = \log_a x + \log_a y$, (b) $\log_a \frac{x}{y} = \log_a x - \log_a y$, (c) $\log_a a = 1$,

(d) $\log_a x = \frac{\log_b x}{\log_b a}$, (e) $\log_a b = \frac{1}{\log_b a}$, (f) $\log_a 1 = 0$

Note **There is no formula for** $\log(m+n)$ **and** $\log(m-n)$ **i.e.,** $\log(m+n) \neq \log m + \log n$ **and** $\log(m-n) \neq \log m - \log n$.

Logarithmic Function : The function which associates each real number x to $\log_a x$ is called logarithmic function. Symbolically, $f(x) = \log_a x$. The logarithm of a positive number is defined as a power to which a base (> 1) must be raised to get that number. Logarithmic functions are inverse to exponential functions. The inverse of an exponential function $y = a^x$ can be written as $y = \log_a x$, where log is abbreviation of logarithm. We note that domain of this function is (0, ∞) and the range is (–∞, ∞).



Since an inverse function is obtained by mere algebraic manipulation, its geometrical properties are similar to the properties of an exponential function. Like an exponential function, a logarithmic function is also monotonic.

Features of Logarithmic Function : The important features of a logarithmic function are as under :

(i) The domain of the function is $(0, \infty)$ and the range is $(-\infty, \infty)$.
(ii) It is a monotonically increasing function.
(iii) The graph of logarithmic function passes through the point (1, 0), it implies that logarithm of unity is always zero.
(i) The logarithm of a number lying between 0 and 1 is negative.
(ii) The logarithm of a negative number is not defined.

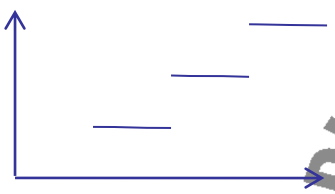
Note $e^{\ln x} = x$ and $\ln e^x = x$.

(xiii) **Signum Function :** The function which is defined as

$$f(x) = \begin{cases} \frac{|x|}{x}, & when\ x \neq 0 \\ x, & when\ x = 0 \end{cases}$$

is known as signum function.

(xiv) **Greatest Integer Function or Step Function:** The Function which associates each real number x to the greatest integer less than or equal to x is called the greatest integer function or step function. Symbolically, $f(x) = [x]$.



(xv) **Transcendental Function :** Functions other than the algebraic functions are called transcendental functions. These include the circular functions defined by $f(x) = \sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, or $\operatorname{cosec} x$,
The exponential functions defined by $f(x) = e^x$ or a^x
The logarithmic functions defined by $f(x) = \log_a x$
And combinations of these and algebraic functions.

(xvi) **Trigonometric Functions :** The functions which associates each real number x to the trigonometric ratios such as sin x, cosine x, tangent x, cotangent x, secant x or cosecant x, are called a trigonometric function.

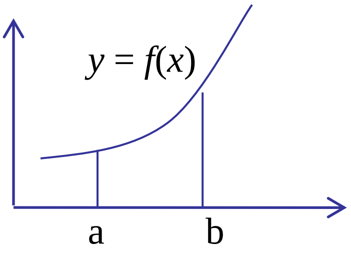
(xvii) **Inverse Trigonometrical Functions :** The inverse trigonometrical functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc. are defined as the inverse of the corresponding trigonometrical functions.

(xviii) **One to One Function :** Let A and B be two non empty sets. A function from A to B is called a one to one function if for each $f(x)$ in range R_f there is one and only one value of x in the domain D_f. In other words, if $f(a) = f(b)$ for elements a and b in the domain of f, then a = b.

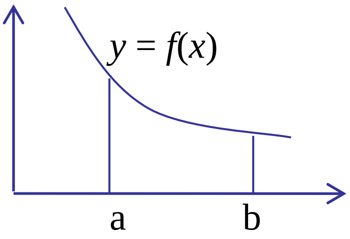
(xix) **Even Function :** A function $f(x)$ is said to be even function if $f(-x) = f(x)$.

(xx) **Odd Function :** A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$.

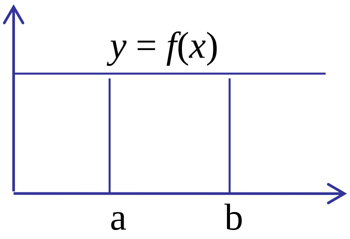
(xxi) **Increasing and Decreasing Functions :** A function $f(x)$ on an interval (a, b) is said to be increasing if $f(a) < f(b)$ for $a < b$ and decreasing if $f(a) > f(b)$ for $a < b$ and constant function if $f(a) = f(b)$.



Increasing function



Decreasing function



Constant function

(xxii) **Inverse Function :** Let $f : A \rightarrow B$ be a one-one onto function, *i.e.*, f be a bijection. Then a function $g : B \rightarrow A$ is called the inverse of the function f iff

$f(a) = b \quad \Leftrightarrow \quad g(b) = a$, for all a $\in$ and b $\in$ B.

Note : **Only one-one and onto functions possess inverse functions.**

<u>Properties of Inverse Function</u>

(a) If f and g are inverses of each other then both are one to one functions.

(b) Let f be a one to one function with the domain A and range B. The function g with domain B and range A is called the inverse of f if

$$g[f(x)] = x \qquad \forall \quad x \in A.$$

(c) The inverse function g of f is denoted by f^{-1}. Thus, $f^{-1}[f(x)] = x \;\; \forall \;\; x \in A$. This means that $f^{-1}[f(x)]$ is an identity function.

(d) $(f^{-1})^{-1} = f$

(e) $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$

(f) If f is differentiable, then f has an inverse on I provided that : either $f'(x) > 0$ for all x in I or $f'(x) < 0$ for all x in I.

(g) Considering function composition helps to understand the notation f^{-1}. Repeatedly composing a function with itself is called iteration. If f is applied n times, starting with the value x, then this is written as $f^{n}(x)$; so $f^{2}(x) = f(f(x))$, etc. Since $f^{-1}(f(x)) = x$, composing f^{-1} and f^{n} yields f^{n-1}, "undoing" the effect of one application of f.

(h) If the function f is differentiable on an interval I and $f'(x) \neq 0$ for each $x \in I$, then the inverse f^{-1} will be differentiable on $f(I)$. Let function g is inverse of function f then

$$g'(y) = \frac{1}{f'(x)}$$

(xxiii) **Involutory Function :** In mathematics, an involution, or an involutory function, is a function f that is its own inverse,

$$f(f(x)) = x$$

for all x in the domain of f.

7. Operations on Functions

(i) **Sum :** Let f and g be two real functions whose domains are respectively D_1 and D_2. Then the sum function denoted by $f + g$ is defined by the rule
$(f + g)(x) = f(x) + g(x)\ \forall\ x \in D$. Where $D = D_1 \cap D_2$ is the common part of the two domains D_1 and D_2 and $D_1 \cap D_2 \neq \phi$.

(ii) **Difference :** Similarly, the difference function denoted by $f - g$ is defined by the rule $(f - g)(x) = f(x) - g(x)\ \forall\ x \in D$. Where $D = D_1 \cap D_2$ is the common part of the two domains D_1 and D_2 and $D_1 \cap D_2 \neq \phi$.

(iii) **Product :** Similarly, the product function denoted by fg is defined by the rule $fg(x) = f(x)g(x)\ \forall\ x \in D$. Where $D = D_1 \cap D_2$ is the common part of the two domains D_1 and D_2 and $D_1 \cap D_2 \neq \phi$.

(iv) **Quotient :** Also, the quotient function denoted by $\frac{f}{g}$ is defined by the rule

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}\ \forall\ x \in D$. Where $D = D_1 \cap D_2 - \{x \mid g(x) = 0\}$ is the common part of the two domains D_1 and D_2 excluding those points at which $g(x) = 0$ and $D_1 \cap D_2 \neq \phi$.

(v) **Scalar Multiplication :** For any real number k, the function kf is defined as
$(kf)(x) = k.f(x)\ \forall\ x \in D$.

Exercise

Topic I : Functions

Q1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) {(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)}

(ii) {(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)}

(iii) {(1,3), (1,5), (2,5)}.

[Ans. Relations (i) & (ii) are functions but relation (iii) is not a function]

Q2. Does the equation $x = |y|$ with $x \geq 0$ represent y as a function of x?
[**Ans.** No]

Q3. Is the rule that assigns to each of the 50 students in a class her marks out of a maximum of 100 marks a function? If yes, is the function one-to-one?
[Ans. Yes, But it is not one-one function]

Q4. The relation f is defined by $f(x) = \begin{cases} x^2 & 0 \leq x \leq 3 \\ 3x & 3 \leq x \leq 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2 & 0 \leq x \leq 2 \\ 3x & 2 \leq x \leq 10 \end{cases}$

Show that f is a function and g is not a function.

Q5. Let f be the subset of $\mathbf{Z} \times \mathbf{Z}$ *defined by* $f = \{(ab, a + b) : a, b \in \mathbf{Z}\}$. Is f a function from $\mathbf{Z}$ to $\mathbf{Z}$? Justify your answer.
[Ans. No]

Q6. Let A = R – {3}, B = R – {1}. Let f : A → B be defined by $f(x) = \dfrac{x-2}{x-3}$. Then

(a) f is bijective (b) f is one-one
(c) f is onto (d) one-one but not onto
[Ans. (a)]

Q7. Given that R denotes the set of real numbers, which of the following mappings is a one-to-one(i.e. injective) function ? **[DSE MA Eco. Entrance 2006]**
(a) $f(x) = \tan x$ where $x \in$ R and $x \geq 0$.
(b) $f(x) = |x|$ where $x \in$ R
(c) $f(x) = 1/x$ where $x \in$ R and $x \geq 0$.
(d) $f(x) = |x|$ where $x \in$ R and $x \geq 0$.
[**Ans. :** (d)]

Q8. Suppose the function f : $R_{++} \to R$ is given by $f(x) = \int_1^x t^{-1}dt$. Consider the following statements : for $x, y \in R_{++}$, **[DSE MA Eco. Entrance 2011]**
$f(x + y) \neq f(x) + f(y)$
$f(xy) = f(x) + f(y)$
In general,
(a) (i) is true and (ii) is false
(b) (i) is false and (ii) is true
(c) Both are true
(d) Both are false
[**Ans. :** (b)]

Q9. Suppose a real valued function f is defined for all real numbers excepting 0, and satisfies the following conditions : $f(xy) = f(x) + f(y)$ for all x, y in the domain. Consider the statements : **[DSE MA Eco. Entrance 2011]**
$f(1) = f(-1) = 0$
$f(x) = f(-x)$ for every x
(a) (i) is true and (ii) is false
(b) (i) is false and (ii) is true
(c) Both are true
(d) Both are false
[**Ans. :** (c)]

Q10. Let N = {1, 2, 3, . . .}. Suppose there is a bijection, *i.e.*, a one-to-one correspondence (an "into" and "onto" mapping), between N and a set X. Suppose there is also a bijection between N and a set Y . Then,

(a) there is a bijection between N and $X \cup Y$
(b) there is a bijection between N and $X \cap Y$
(c) there is no bijection between N and $X \cap Y$
(d) there is no bijection between N and $X \cup Y$

[**Ans. :** (a)] **[DSE MA Eco. Entrance 2015]**

Q11. If $f(x + 2y, x - 2y) = xy$, then $f(x, y)$ equals **[ISI MS QE 2015]**

(a) $\dfrac{x^2 - y^2}{8}$ (b) $\dfrac{x^2 - y^2}{4}$ (c) $\dfrac{x^2 + y^2}{4}$ (d) None of these

[**Ans. :** (a)]

Q12. How many onto functions are there from a set A with m > 2 elements to a set B with 2 elements? **[ISI MS QE 2017]**

(a) 2^m (b) $2^m - 1$ (c) $2^{m-1} - 2$ (d) $2^m - 2$

[**Ans. :** (d)]

Topic II : Domain and Range of Functions

Q1. For the following functions what are the domain and range of f?

(a) $f(x) = \dfrac{x}{x^2 + 1}$, (b) $f(x) = (x-1)^{0.5}$ (c) $\dfrac{1}{|x|}$

(d) $\dfrac{1}{|x|}$, $x \in [-1, 5]$ (e) $f(x) = \sqrt{\dfrac{(3-x)^2}{2+x}}$ (f) $f(x) = \sqrt{\dfrac{-1}{x+1}}$

[Ans. (a) Domain : $x \in$ R, Range : $\dfrac{-1}{2} \le f(x) \le \dfrac{1}{2}$, **(b)** Domain : $x \ge 1$, Range : $f(x) \ge 0$]

Q2. The domain of the function $f(x) = \sqrt{x-1} + \sqrt{6-x}$, is

(a) $[1, \infty)$ (b) $(-\infty, 6)$ (c) $[1, 6]$ (d) none of these

[**Ans.** (c)]

Q3. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$, is

(a) $[0, \infty)$ (b) $(-\infty, 0)$ (c) $(-\infty, 0]$ (d) $[1, \infty)$

[**Ans.** (b)]

Q4. The domain of the function $f(x) = \sqrt{x^2 - 1} - \log\left(\sqrt{1-x}\right)$, $x \ge 0$ is

(a) $(-\infty, -1)$ (b) $(-1, 0)$ (c) null set (d) none of these

[**Ans.** (a)] **[ISI MS QE 2015]**

Q5. Find the domain of the following functions :

(i) $f(x) = |x|$

(ii) $f(x) = \dfrac{x-3}{x+1}$

(iii) $f(x) = \dfrac{x^3 - 5x + 3}{x^2 - 1}$

(iv) $f(x) = \dfrac{a^{3x} - b^{4x}}{x}$

(v) $f(x) = \dfrac{1}{x+3}$

(vi) $f(x) = \sqrt{2x+4}$

(vii) $f(x) = \sqrt{5-x}$

(viii) $f(x) = \dfrac{2x-1}{x^2 - x}$

(ix) $f(x) = \sqrt{\dfrac{x-1}{(x-2)(x-3)}}$

(x) $f(x) = \sqrt{x+1} + \dfrac{1}{\sqrt{x+1}}$

[Ans. (i) R, **(ii)** R – {–1}, **(iii)** R – {–1, 1}, **(iv)** R – {0},
(v) R – {–3}, **(vi)** $x \geq -2$, **(vii)** x ≤ 5, **(viii)** R – {0, 1}, **(ix)**
$-3 < x \leq 1$ or $x > 2$, **(x)** x > 1]

Q6. Find the range of the following functions :

(i) $f(x) = x^2$

(ii) $f(x) = \sqrt{9 - x^2}$

(iii) $f(x) = \dfrac{2+x}{2-x}$

[Ans.(i) [0, ∞),**(ii)** [0, 3], **(iii)** R – {–1},]

Q7. Calculate the domain of the following function :

$f(x) = \dfrac{1}{x - \sqrt{x+2}}$

[Ans. $x \geq -2$ and $x \neq 2$]

Q8. Determine the domain on which the equation $y^6 = x$ defines y as a function of x.
[Ans. $x \geq 0$]

Q9. Find the domain and range of the function :

$$y = \ln\left(\frac{x^2 - 4}{x^2 + 4}\right)$$

[Ans. $x \leq -2$ or $x \geq 2$]

Q10. Find the domain and range of $\sqrt{5 - 4x - x^2}$.
[Ans. Domain : $-5 \leq x \leq 1$, Range : $0 \leq y \leq 3$]

Q11. Let

$$f(x) = \begin{cases} x^2 - 4x + 3 & x < 3 \\ x - 4 & x \geq 3 \end{cases} \text{ and } g(x) = \begin{cases} x - 3 & x < 3 \\ x^2 + 2x + 2 & x \geq 3 \end{cases}$$

Describe the function f / g and find its domain.

Q12. Consider the function f mapping points of the plane into the plane, defined by $f(x, y) = (x - y, x + y)$. The range of this function is

(a) the 45 degree line

(b) a ray through the origin but not the 45 degree line

(c) the entire plane

(d) the first and third quadrants **[DSE MA Eco. Entrance 2011]**

[**Ans. :** (c)]

Topic III : Types of Functions

Q1. Let A = R – {2} and B = R – {1}. If $f : A \to B$ is defined by $f(x) = \frac{x-1}{x-2}$ for all $x \in A$, show that f is one-one.

Q2. Determine whether the following functions are one-one or not.

(i) $f : R \to R$ given by $f(x) = x^3 + 4 \ \forall \ x \in R$.

(ii) $f : Z \to Z$ given by $f(x) = x^2 + 1 \ \forall \ x \in Z$.

[Ans.(i) Yes, **(ii)** No]

Q3. Determine whether the following functions are even or odd :

(i) $y = 3x^2 + 4$, (ii) $y = \frac{x^2 - 1}{x^2 + 4}$, (iii) $y = x^3$, (iv) $y = 2x^4 - x^2 + 1$

[Ans.(i) Even, **(ii)** Even, **(iii)** Odd, **(iv)** Even]

Q4. $f(x) = x^2 + kx + 1$ for all x, if $f(x)$ is an even function, find k.

[Ans. $k = 0$]

Q5. The expenditure of household on consumer goods (C) is related to the household income (Y) in the following way. When the household's income is Rs. 1000 the expenditure on consumer goods is Rs. 900, and whenever income is increased by Rs. 100, the expenditure on consumer goods is increased by Rs. 80. Express the expenditure on consumer goods as a function of income, assuming a linear relationship.

[Ans. $C = 100 + 0.8Y$]

Q6. Determine the numbers a, b, c such that the function y = ax^2 + bx + c fits to the data points (–1, 7), (0, 4) and (2, 6). Hence express the quadratic function.

[Ans. $a = \frac{4}{3}, b = -\frac{5}{3}, c = 4, \quad y = \frac{4}{3}x^2 - \frac{5}{3}x + 4$]

Q7. The curve $f(x) = x^2 + 4x + 5$ is symmetric about the line x = k. Find k.

[Ans. k = -2]

Q8. Find the maximum and minimum point of each of the following parabolas :

(i) $f(x) = x^2 - 4x + 3$, (ii) $f(x) = -2x^2 + 40x - 600$

[Ans.(i) (2, –1), **(ii)** (10, –400)]

Q9. Solve the following equations :

(i) $x^4 - 5x^2 + 4 = 0$ (ii) $x^6 - 9x^3 + 8 = 0$

[Ans. (i) $x = \pm 1, x = \pm 2$]

Q10. A model occurring in the theory of efficient loan markets involves the function :

$$U(x) = 72 - (4 + x)^2 - (4 - rx)^2$$

Where r is a constant. Find the value of x for which $U(x)$ attains its largest value.

Topic IV : Exponential and Logarithmic Function

Q1. Prove that $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$.

Q2. Find the values of following

(i) ln1 (ii) ln$e$ (iii) ln(1/e) (iv) ln4 (v) ln(-1)

[Ans.(i) 0, **(ii)** 1, **(iii)** -1, **(iv)** 1.386, **(v)** Not defined]

Q3. Solve the following equations for x :

(i) $5e^{-3x} = 16$ (ii) $A\alpha e^{-\alpha x} = k$ (iii) $e^x + e^{-x} = 2$ (iv) $3^x = 8$

(v) ln$x$ = 3 (vi) $\ln(x^2 - 4x + 5) = 0$ (vii) $\ln[x(x - 2)] = 0$

(viii) $\ln(\sqrt{x} - 5) = 0$

[Ans.(i) $x = \frac{1}{3}\ln\frac{5}{16}$, **(ii)** $x = \frac{1}{\alpha}\ln\frac{A\alpha}{k}$, **(iii)** $x = 0$, **(iv)** $x = \frac{\ln 8}{\ln 3}$,

(v) $x = e^3$, **(vi)** $x = 2$, **(vii)** $x = 1 \pm \sqrt{2}$, **(viii)** $x = 36$]

Q4. Solve for x : $\log_x(8x - 3) - \log_x 4 = 2$.

[Ans. $x = \frac{1}{2}$ or $x = \frac{3}{2}$]

Q5. Find the domains of the following functions :

(i) y = $ln(1 - x)$ (ii) y = $ln(4 - x^2)$

(iii) $y = \ln\left[\frac{x-1}{x+1}\right]$

[Ans.(i) $x < 1$, **(ii)** $-2 < x < 2$, **(iii)** $x < -1$ or $x > 1$]

Q6. Solve the following inequalities :

(i) $\ln x \le -1$ (ii) $\ln(x^2 - x - 1) \ge 0$ (iii) $\ln x + \ln(x-3) \le \ln 4$

[Ans.(i) $0 < x \le 1/e$, **(ii)** $x \le -1$ or $x \ge 2$, **(iii)** $3 < x \le 4$]

Q7. Transform the following function to its natural logarithmic form :
$y = 8\log_2(2x)$.

[Ans. $y = \dfrac{8\ln(2x)}{\ln 2}$]

Topic V : Composition of Functions

Composition of Functions : If f and g are two real functions, whose domains are respectively D_1 and D_2, the composition of f with g, denoted by $f\,o\,g$ is the function defined by $(f\,o\,g)(x) = f\{g(x)\}$.

The Domain of $f\,o\,g$ is the set of all those $x \in D_2$ (Domain of g) for which $g(x) \in D_1$ (Domain of f). But if the range of g is a subset of domain of f, then domain of $f\,o\,g$ is same as the domain of g.

Q1. Let $f(x) = \dfrac{x+\sqrt{3}}{1-\sqrt{3}x}$, for all $x \ne \dfrac{1}{\sqrt{3}}$. What is the value of $f(f(x))$?

(a) $\dfrac{x-\sqrt{3}}{1+\sqrt{3}x}$ (b) $\dfrac{x^2+2\sqrt{3}x+3}{1-2\sqrt{3}x+3x}$

(c) $\dfrac{x+\sqrt{3}}{1-\sqrt{3}x}$ (d) $\dfrac{x+\sqrt{3}}{1-\sqrt{3}x}$

[ISI MS QE 2016]

Q2. Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ find $fofof$.

[**Ans.** $\dfrac{x}{\sqrt{1+3x^2}}$]

Q3. Let $f(x) = \dfrac{1}{1-x}$ then $fofof$ is an identity function on its domain.

Q4. If $f(x) = e^x$ and $g(x) = \log x$, show that $fog = gof$.

Q5. If $f(x) = e^x$ and $g(x) = \log x$, then determine the values of the following :

(i) $(f+g)(1)$ (ii) $(fg)(1)$ (iii) $(fog)(1)$

[Ans. (i) e, **(ii)** 0, **(iii)** 1]

Q6. If $f(x) = [x]$ and $g(x) = |x|$, then determine the values of the following :

(i) $(g\,o\,f)\left(-\frac{5}{3}\right)-(f\,o\,g)\left(-\frac{5}{3}\right)$ (ii) $(g\,o\,f)\left(\frac{5}{3}\right)-(f\,o\,g)\left(\frac{5}{3}\right)$

[Ans. (i) 1, **(ii)** 0]

Q7. If $f(x) = a^x$, prove that

(i) $f(x+y) = f(x).f(y)$ (ii) $f(x-y)=\dfrac{f(x)}{f(y)}$

Q8. let $A(x)=\dfrac{e^x-e^{-x}}{2}$ and $B(x)=\dfrac{e^x+e^{-x}}{2}$, show that :

(i) $B(x + y) = A(x)A(y) + B(x)B(y)$, (ii) $A(2x) = 2A(x)B(x)$

Q9. Let $f(x)=\dfrac{x+1}{x+2}, x\neq -2$ and $g(x)=x^2$, then describe the following

(i) fg (ii) $\dfrac{f}{g}$ (iii) fog (iv) gof

[Ans. (i) $\dfrac{x^3+x^2}{x+2}$, **(ii)** $\dfrac{x+1}{x^3+2x^2}$, **(iii)** $\dfrac{x^2+1}{x^2+2}$, **(iv)** $\left(\dfrac{x+1}{x+2}\right)^2$]

Q10. Let $f(x)=x^2+1$ and $g(x)=3x+4$. Determine each of the following :

(i) fog (ii) gof

[Ans. (i) $9x^2+24x+17$, **(ii)** $3x^2+7$]

Q11. If $f(x)=e^{2x}$ and $g(x)=\log\sqrt{x}$ find $fog\left(\frac{3}{2}\right)$

[Ans. 3/2]

Q12. Let $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$. Find *fof*.

[Ans. $fof(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$]

Q13. Let $f(x)=\dfrac{ax+b}{cx-a}$, where a, b, and c are constants, and $c \neq 0$. Assuming that $x \neq a/c$, show that $f\left(\dfrac{ax+b}{cx-a}\right)=x$.

Topic VI : Inverse of a Function

Q1. Show that function $f(x) = 3x + 2$ and $g(x) = \frac{x-2}{3}$ are inverses of each other.

Q2. Show that function $f(x) = \sqrt{x-2}, x \geq 2$ and $g(x) = x^2 + 2, x \geq 0$ are inverses of each other.

Q3. Let $f(x) = 2x + 3$. Find f^{-1}.

Q4. Let $f(x) = \sqrt{9 - x^2} \quad 0 \leq x \leq 3$. Find f^{-1}.
[Ans. $f^{-1}(x) = \sqrt{9 - x^2} \quad 0 \leq x \leq 3$]

Q5. Find the inverse function of $f(x) = \frac{x-1}{x+1}, x \neq -1$ and verify that fof^{-1} is an identity function.
[Ans. $f^{-1}(x) = \frac{1+x}{1-x}, x \neq 1$.]

Q6. Show that $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and $x = \frac{1}{2}\log\frac{1+y}{1-y}$ are inverse functions.

Q7. If $f(x) = x^3 - 3x^2 + x$ and g is the inverse of f, in a small open interval with centre 3, then find $g'(3)$.
[Ans. 1/10]

Q8. Let $h(x) = \frac{x-1}{-x+3}, x \neq 3$
(i) Find the inverse of h.
(ii) Find the range of h.
[Ans.(i) $h^{-1}(x) = \frac{3x+1}{x+1}$, **(ii)** $y \in$ R]

Q9. Let f be defined on [0, 5] by $f(x) = x^2$.
Show that f has an inverse function $g(x)$. Draw graphs of f and g. How are the two graphs related?

Q10. Let $f(x) = x^2 + 3$, $x \geq 0$. Does this function have an inverse? Sketch both $f(x)$ and its inverse (if it exists). Show both curves in one graph.
[Ans. No, it does not have an inverse]

Q11. Do the following functions have an inverse ? Give reasons for your answer :

(i) $f(x) = 8x^7 + 4x^5 + 9x^3 + 21x$ (ii) $f(x) = 10 - x^2$

Q12. Let f be defined on [0, 1] by $f(x) = 2x^2 - x^4$.

(i) Find the range of the function.

(ii) Show that f has an inverse function g, and find a formula for g.

Topic VII : Operations on Functions

Q1. Determine which of the functions are increasing, decreasing or constant functions :

(i) $f(x) = 5x^2 + 3$, (ii) $f(x) = -2x + 4$, (iii) $f(x) = 5$

[Ans.(i) increasing, **(ii)** decreasing, **(iii)** constant]

Q2. For each real number x the value $y = [x]$ is the largest integer that is less than or equal to x. For what values of x is $[x] = 0$? Graph y for these values of x.

Q3. Let $f : R^2 \to R$ be a function. For every $x, y, z \in R$, we know that $f(x, y) + f(y, z) + f(z, x) = 0$. Then, for every $x, y \in R^2$, $f(x, y) - f(x, 0) + f(y, 0) =$

(a) 0 (b) 1 (c) -1 (d) None of the above.

[**Ans. :** (c)] **[ISI MS QE 2016]**

Trigonometric Functions

Measurement of an angle

Sexagesimal system: In this system unit of measurement is **degree.**

1 right angle = 90^0 (90 degrees)

$10^0 = 60'$ (60 minutes)

$1' = 60''$(60 seconds)

Circular measure of an angle In this system unit of measurement is a **radian.**

One radian is defined as an angle formed at the centre of circle by an arc whose length is equal to the radius of circle.

Radian is a constant angle. It is equal to $\frac{2rt.\angle s}{\pi}$. It is denoted as 1^c.

We have $\pi^0 = 180^0 = 2rt\angle s$.

Conversion of angles in degrees into angles in radians

$\theta^c = (\theta^0 * \pi)/180$

Relation between Trigonometric ratios

$$\sin\theta = \frac{Perpendicular}{Hypotenuse} \qquad \cos ec\theta = \frac{Hypotenuse}{Perpendicular}$$

$$\cos\theta = \frac{Base}{Hypotenuse} \qquad \sec\theta = \frac{Hypotenuse}{Base}$$

$$\tan\theta = \frac{Perpendicular}{Base} \qquad \cot\theta = \frac{Base}{Perpendicular}$$

Value of T-ratios

θ	0	30^0	45^0	60^0	90^0
sin θ	0	½	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0
tan θ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
cosec θ	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1
sec θ	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
cot θ	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0

Trigonometric Identities

(i) $\sin^2 A + \cos^2 A = 1$

$\Rightarrow \quad \sin^2 A = 1 - \cos^2 A$

$\Rightarrow \quad \cos^2 A = 1 - \sin^2 A$

(ii) $\sec^2 A = 1 + \tan^2 A$

$\Rightarrow \quad \sec^2 A - 1 = \tan^2 A$

$\Rightarrow \quad \sec^2 A - \tan^2 A = 1$

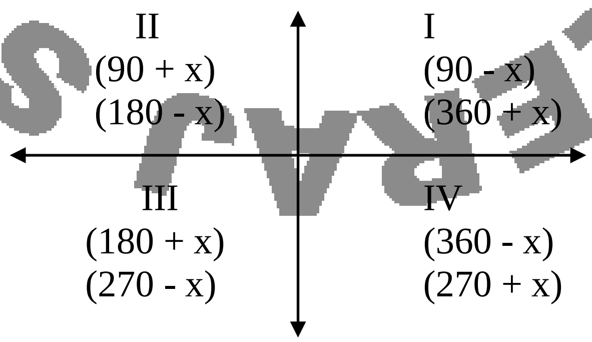
(iii) $\text{cosec}^2 A = 1 + \cot^2 A$

$\Rightarrow \quad \text{cosec}^2 A - 1 = \cot^2 A$

$\Rightarrow \quad \text{cosec}^2 A - \cot^2 A = 1$

Quadrant Rule

1.



2.

II	I
sin, cosec +ve	ALL +ve
tan, cot +ve	**cos, sec** +ve
III	IV

Rule

In **FIRST** quadrant **ALL** trigonometric ratio's are positive,
In **SECOND** quadrant only **sin and cosec** are positive, other's are negative
In **THIRD** quadrant only **tan and cot** are positive, other's are negative and
In **FOURTH** quadrant only **cos and sec** are positive, other's are negative.

Aid to memory

After	**S**chool	**T**o	**C**ollege
|	|	|	|
All	**S**in	**T**an	**C**os

3. At $\pi/2$ and $3\pi/2$

sin changes to **cos**	&	**cos** changes to **sin**
tan changes to **cot**	&	**cot** changes to **tan**
sec changes to **cosec**	&	**cosec** changes to **sec**

At π and 2π **NO CHANGE**

T ratio's of sum and difference of two angles

sin (A + B) = sin A cos B + cos A sin B
sin (A – B) = sin A cos B – cos A sin B
cos (A + B) = cos A cos B – sin A sin B
cos (A – B) = cos A cos B + sin A sin B

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$ $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

sin(A + B + C) = sin A.cos B.cos C + cos A.sin B.cos C + cos A.cos B.sin C – sin A.sin B.sin C cos(A + B + C) = cos A.cos B.cos C – cos A.sin B.sin C – sin A.cos B.sin C – sin A.sin B.cos C

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Representation of product of T-ratio's as sum or difference

2 sin A cos B = sin (A + B) + sin (A – B)
2 cos A sin B = sin (A + B) – sin (A – B)
2 cos A cos B = cos (A + B) + cos (A – B)
2 sin A sin B = cos (A – B) – cos (A + B).

Representation of sum or difference of T – ratio's as product

sin A + sin B = $2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

sin A – sin B = $2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

cos A + cos B = $2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

cos A – cos B = $-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B) = \cos^2 B - \cos^2 A$
$\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A$

tan C + tan D = tan(C + D)[1 – tan C . tan D] = $\frac{\sin(C+D)}{\cos C.\cos D}$

tan C – tan D = tan(C – D)[1 + tan C . tan D] = $\frac{\sin(C-D)}{\cos C.\cos D}$

cot C + cot D = $\frac{\cot C.\cot D-1}{\cot(C+D)}$ = $\frac{\sin(C-D)}{\sin C.\sin D}$

cot C – cot D = $\frac{\cot C.\cot D+1}{\cot(C-D)}$ = $\frac{-\sin(C-D)}{\sin C.\sin D}$

T ratio's of multiple angles

sin 2A = 2 sin A cos A = $\frac{2\tan A}{1+\tan^2 A}$ tan 2A = $\frac{2\tan A}{1-\tan^2 A}$

cos 2A $= \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
$= 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

$1 - \cos 2A = 2\sin^2 A$ $1 + \cos 2A = 2\cos^2 A$

$$1-\cos mA = 2\sin^2\frac{mA}{2} \qquad 1+\cos mA = 2\cos^2\frac{mA}{2}$$

sin 3A = 3 sin A – 4 sin^{3}A $\Rightarrow$ $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

cos 3A = 4 cos^3 A – 3 cos A $\Rightarrow$ $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

tan 3A = $\frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

T ratio's of sub-multiple angles

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \cot\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \qquad \tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1-2\sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$1-\cos A = 2\sin^2\frac{A}{2} \qquad 1+\cos A = 2\cos^2\frac{A}{2}$$

T ratio's of angle of measure 18^0 and its multiples

$$\sin 18^0 = \frac{\sqrt{5}-1}{4} \qquad \cos 18^0 = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\sin 36^0 = \frac{\sqrt{10-2\sqrt{5}}}{4} \qquad \cos 36^0 = \frac{\sqrt{5}+1}{4}$$

General Solution of Trigonometric equations:

(i) If $\sin\theta = 0$, then $\theta = n\pi, n \in I$

(ii) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}, n \in I$

(iii) If $\tan\theta = 0$, then $\theta = n\pi, n \in I$

(iv) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha, n \in I$

(v) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha, n \in I$

(vi) If $\tan\theta = 0$, then $\theta = n\pi + \alpha, n \in I$

(vii) If $\sin^2\theta = \sin^2\alpha$, then $\theta = n \pm \alpha, n \in I$

(viii) If $\cos^2\theta = \cos^2\alpha$ then $\theta = n\pi \pm \alpha, n \in I$

(ix) If $\tan^2\theta = \tan^2\alpha$ then $\theta = n\pi \pm \alpha, n \in I$