Chapter 1

Constrained Optimization

Learning Objectives :

After learning this chapter you will understand :

- **Two Variables, One Equality Constraint.**
- **Lagrange Multiplier Method.**
- **Sufficient Conditions for Constrained Optimization.**
- **Economic Interpretations of Lagrange Multipliers.**
- **Envelope Theorem.**

Basic Concepts

1. **Constrained Optimization :** Constrained optimization is a type of mathematical optimization problem where the goal is to find the best solution for a given objective function, subject to a set of constraints. The objective function is the function that needs to be either maximized or minimized, while the constraints are a set of limitations or restrictions that must be satisfied by the solution.

Mathematically, a problem of maximizing (or minimizing) a function $f(x, y)$ when x and y are restricted to satisfy an equation $g(x, y) = c$. Here, $f(x, y)$ is called the objective function and $g(x, y) = c$ is called the constraint. So, we write the problem as

Objective function Max (or Min) $f(x, y)$

Subject to $g(x, y) = c$

Constrained optimization plays a crucial role in various economic scenarios. Here are some economic examples of constrained optimization:

(i) **Consumer Choice and Utility Maximization:** In microeconomics, consumers aim to maximize their utility (objective) subject to a budget constraint. The consumer must decide how to allocate their income to purchase goods and services while staying within their budget limit. The optimal choice involves maximizing utility while considering the prices of goods and the consumer's income as given. In mathematical terms, the consumer faces the problem of choosing quantity of two goods (x, y) in order to maximize utility $u(x, y)$ under a given budget constraint $p_1x + p_2y = m$.

(ii) **Production Optimization:** In a manufacturing setting, a company may want to maximize its production output (objective) subject to constraints such as limited resources (e.g., raw materials, labor, machines), production capacity, and environmental regulations. The goal is to find the optimal allocation of resources that maximizes production while adhering to these constraints. In mathematical terms, therefore, the producer faces the problem of choosing quantity of two inputs (L, K) in order to maximize Output $Q = F(L, K)$ under a given cost constraint $p_L L + p_K K = C$.

2. **Two Variables, One Equality Constraint :** Consider the problem of maximizing (or minimizing) a function $f(x, y)$ when x and y are restricted to satisfy an equation $g(x, y) = c$. The value of objective function $f(x, y)$ is optimum under the constraint $g(x, y) = c$ when the constraint curve touches the level curve (without intersecting) a level curve of f. The slope of level curve $f(x, y) = c$ is given by $\frac{dy}{dx} = -\frac{f_1'(x,y)}{f_2'(x,y)}$ and slope of the tangent to $g(x, y) = c$ is $-\frac{g_1'(x,y)}{g_2'(x,y)}$. The condition for optimizing the objective function is

$$-\frac{f_1'(x,y)}{f_2'(x,y)} = -\frac{g_1'(x,y)}{g_2'(x,y)} \quad \Rightarrow \quad \frac{f_1'(x,y)}{f_2'(x,y)} = \frac{g_1'(x,y)}{g_2'(x,y)} \quad \ldots\ldots\ldots(i)$$

On solving equation (i) along with $g(x, y) = c$ determines the two unknowns x and y which optimize $f(x, y)$ subject to $g(x, y) = c$.

Exercise 1

Q1. Find the only possible solution to the problem
max xy subject to $2x + y = m$

Q2. Find the only possible solution to the problem
max $f(x, y) = x^{\alpha}y^{\beta}$ subject to $px + y = m$

Q3. Find the only possible solution to the problem
max (min) $f(x, y) = x + y$ subject to $x^2 + 3xy + 3y^2 = 3$

Answers of Exercise 1

1. $x = m/4$, $y = m/2$, **2.** $x = \frac{\alpha}{\alpha+\beta}.\frac{m}{p}, y = \frac{\beta}{\alpha+\beta}.\frac{m}{1}$

3. $f(x, y)$ is maximum at x = 3, y = -1 & maximum value of $f(x, y)$ is 2
$f(x, y)$ is minimum at x = -3, y = 1 & minimum value of $f(x, y)$ is -2

Basic Concepts

1. **Lagrange Multiplier Method :** The Lagrange multiplier method is a powerful technique used in constrained optimization problems to find the extreme values (maximum or minimum) of a function subject to one or more constraints.

In the Lagrange multiplier method for constrained optimization with two variables and one constraint, we aim to find the extreme values (maximum or minimum) of a function $f(x, y)$ subject to a single constraint $g(x, y) = c$. The method involves introducing a Lagrange multiplier λ to construct the Lagrange function and then finding the critical points to determine the extreme values. Here, $f(x, y)$ is called the objective function and $g(x, y) = c$ is called the constraint.

2. **Optimization Under Constraints by Lagrange's Multiplier Method :** When an objective function $Z = f(x, y)$ is to be optimized subject to a single constraint $g(x, y) - k = 0$, the following steps are required :

(i) Construct an Auxiliary Function (or Lagrange Function) V of three variables defined as :
$V(x, y, \lambda) = f(x, y) - \lambda.[g(x, y) - k]$, where λ is known as Lagrange Multiplier.

(ii) Find the partial derivatives of V w.r.t x, y and λ and put them equal to zero, such that $V_x = 0$, $V_y = 0$ and $V_\lambda = 0$ and solve these equations to get the values of x and y, *eg.*, (a, b).

(iii) Find the partial derivatives V_{xx}, V_{xy}, V_{yy} and g_x, g_y.

(iv) Find the value of the Bordered Hessian Determinant

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & V_{xx} & V_{xy} \\ g_y & V_{xy} & V_{yy} \end{vmatrix}$$

(v) If $|\bar{H}_2| > 0$ then function is maximum at (a, b) and if $|\bar{H}_2| < 0$ then function is minimum at (a, b).

Exercise 2

Q1. Find the maxima or minima of the function $z = 5x^2 + 6y^2 - xy$ subject to $24 = x + 2y$.

Q2. Find the maxima or minima of the function $z = 12xy - 3y^2 - x^2$ subject to $x + 2y - 24 = 0$.

Q3. Solve the problem :

Max. (Min.) $f(x, y) = x^2 + y^2$ subject to $g(x, y) = x^2 + xy + y^2 = 3$

Q4. Use the method of Lagrange multiplier to find :

(i) the minimum value of $x + y$, subject to the constraint $xy = 1$;

(ii) the maximum value of xy, subject to the constraint $x + y = 16$.

Comment on the geometry of each solution. **[Eco. (H) 2008]**

Q5. Find the maximum and minimum of :

$f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$. **[Eco. (H) 2011]**

Q6. Maximize $f(x, y) = 100\ln x + 50\ln y$, where $x > 0$, $y > 0$

Subject to $3x + y = 10$. **[Eco. (H) II Sem. 2015(ER)]**

Q7. Maximize the function Z = xy, subject to the constraint $x + y = 1$. Compare it with minimizing the function $g = (x + y)$, subject to the constraint $xy = ¼$.

[Eco. (H) II Sem. 2013]

Q8. Solve the optimization problem using Lagrange's method :

Max $f(x) = x_1^{0.5} x_2^{0.2}$ subject to $100 - 2x_1 - 3x_2 = 0$. **[BBE II Sem. 2013]**

Q9. Solve the following problem by Lagrangean method. **[BBE II Sem. 2014]**

Max $12x\sqrt{y}$ Subject to $3x + 4y = 12$.

Q10. Find the maximum and minimum values of :

$f(x, y) = x + y$ Subject to the constraint : $g(x, y) = x^2 + y^2 = 4$

Write down the Lagrangean function for the problem and solve the necessary conditions for the optimal solutions. Find the determinant, D(x, y) and check the second order condition. Explain the optimization solution geometrically by drawing the level curves of $f(x, y)$ together with the graph of the constraint.

[Eco. (H) II Sem. 2012]

Q11. Solve, using Lagrangian method, the following problem :**[Eco. (H) II Sem. 2014]**

$$\underset{x, y}{Min}\ f(x, y) = x + y$$

Subject to

$$x^2 - y = 1$$

Also explain the problem geometrically by drawing appropriate level curves for f together with the graph of the parabola $x^2 - y = 1$. Does the associated maximization problem have a solution? Justify your answer.

Q12. Find the maximum and minimum values that the function $f(x, y) = xy$ takes on the constraint : **[Eco. (H) II Sem. 2016]**

$$\frac{x^2}{8} + \frac{y^2}{8} = 1$$

Q13. Use the Lagrangean method to find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$. **[Eco. (H) II Sem. 2017]**

Questions from Recent Eco (H) Examination

Q14. A point moves on the curve $x^2 + y^2 = 100$. At what point is its distance from the point (x, y) = (10, 8) minimum? If the constant 100 in the equation of the curve were to be increased by one unit, what is the instantaneous effect on the minimum distance. **[Eco. (H) II Sem. 2019]**

Q15. Use the method of Lagrange multiplier to find the following two –

(i) The minimum value of $x + y$ subject to the constraint $x^{1/2}y^{1/2} = 2$

(ii) The maximum value of $x^{1/2}y^{1/2}$ subject to the constraint $x + y = 4$.

[Eco. (H) II Sem. 2021]

Answers of Exercise 2

1. Min. z = 612 at $x = 6$, $y = 9$,

2. Max. z = 528 at $x = 9$, $y = 7$,

3. Minimum at (1, 1), maximum at (–1, –1),

4.(i) Minimum at (–1, –1) and Maximum at (1, 1); **(ii)** Maximum at (8, 8),

5. Maximum at (10, –6), Minimum at (–10, 6) ,

14. At $\left(\frac{50}{\sqrt{41}}, \frac{40}{\sqrt{41}}\right)$ the distance is minimum.

Basic Concepts

1. **Economic Applications of Lagrange Multiplier :** Lagrange multipliers are an essential tool in optimization problems where certain constraints must be satisfied. They have numerous economic applications, especially in microeconomics and econometrics. Here are some economic applications of Lagrange multipliers:

(i) **Utility Maximization:** In microeconomics, consumers aim to maximize their utility (satisfaction) subject to a budget constraint. Lagrange multipliers help find the optimal combination of goods and services that maximize utility while considering the budget constraint.

(ii) **Profit Maximization:** Firms aim to maximize their profits by choosing the optimal combination of inputs (e.g., labor and capital) while considering production constraints. Lagrange multipliers help identify the optimal input levels for profit maximization.

(iii) **Cost Minimization:** Firms seek to minimize production costs while achieving a certain level of output. Lagrange multipliers can be used to find the optimal combination of inputs that minimize costs under production constraints.

2. **Economic Interpretation of λ :** The Lagrange Multiplier 'λ' is referred to as a **shadow price** or **marginal value** imputed to a unit of resource.

Consider the problem

Max $f(x, y)$ subject to $g(x, y) = c$

Suppose x^* and y^* are the values of x and y that solve this problem. In general x^* and y^* depend on c. we assume that $x^* = x^*(c)$ and $y^* = y^*(c)$ are differentiable functions of c. The associated value f^* of $f(x, y)$ is then also a function of c, with

$f^*(c) = f(x^*(c), y^*(c))$

Here $f^*(c)$ is called the optimal value function for the problem.

***The Lagrange multiplier* λ =** λ(c) is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant c.

Such that

$$\frac{df^*(c)}{dc} = \lambda(c) \qquad \Rightarrow \qquad df^*(c) = \lambda . dc$$

In particular dc is a small change in c then

$$f^*(c + dc) - f^*(c) \approx \lambda(c). dc$$

For some specific economic problems the meaning of λ is discussed as under :

(i) **Cost Minimisation :** Let $Q = f(L, K)$ be the production function of a firm, where K and L denote the units of capital and labour respectively and Q denotes the level of output. Let r and w be the per unit price of capital and labour respectively. If the producer wants to minimize his cost subject to a given level of income or output target, the lagrange function can be written as $V = rK + wL - \lambda[f(L, K) - Q]$. Here, λ represents the marginal cost of the firm.

(ii) **Revenue or Profit Maximization :** Let $Q = f(L, K)$ be the production function of a firm, where K and L denote the units of capital and labour respectively and Q denotes the level of output. Let r and w be the per unit price of capital and labour respectively. If the producer wants to maximize his revenue subject to a given level of Cost (C), the lagrange function can be written as $V = f(L, K) - \lambda[rK + wL - C]$. Here, λ represents marginal productivity.

(iii) **Equilibrium of the Consumer :** If we want to find the equilibrium condition of a consumer with utility function $U = f(x, y)$ and the budget constraint $xp_x + yp_y = M$, where the symbols have their usual meanings, the lagrange function can be written as $V = U - \lambda[xp_x + yp_y - M]$. Here, the lagrange multiplier 'λ' represents the marginal utility of money income.

Note : If the consumer wants to minimize M subject to a given level of satisfaction, the Lagrange function can be written as $V = xp_x + yp_y - \lambda[f(x, y) - U]$.

Exercise 3

Minimising Cost

Q1. A producer desires to minimize his cost of production C = 2L + 5K where L and K are inputs, subject to the production function Q = LK. Using Lagrange Multiplier Method find the optimum combination of inputs if total output is 40 units.

Q2. The production function of a firm is $Q = K^{1/2}L^{1/2}$ and prices of capital and labour are fixed at Rs. r and w respectively.

(i) Find the cost minimizing combination of capital and labour.

(ii) Derive the demand functions of capital and labour.

(iii) Derive the cost function of the firm.

Q3. If $X = A\sqrt{LK}$ is the production function, find the amounts of the factors used at given prices P_L and P_K to produce an output X at smallest cost. In case of Pure competition on the market for X with demand law $x = \beta - \alpha . p$. Show that the demand for the factors are :

$L = \frac{\beta}{A} . \sqrt{\frac{P_K}{P_L}} - \frac{2\alpha P_K}{A^2}$ and $K = \frac{\beta}{A}\sqrt{\frac{P_L}{P_K}} - \frac{2\alpha P_L}{A^2}$. **[Eco. (H) 1996]**

Maximising Output or Profit

Q4. Optimize the following CES production function subject to the given constraint :

$q = 80[0.4K^{-0.25} + 0.6L^{-0.25}]^{-1/0.25}$ subject to 5K + 2L = 150.**[BBE II Sem. 2013]**

Q5. A firm's production function is given by :

$Q = 3\sqrt{L} + 5\sqrt{K}$, where Q is the output of the firm and L and K are the two inputs, labour and capital. The wage rate is Rs. 6 and price of capital is Rs. 4 per unit. The price of output is Rs. 200.

(a) If the firm is prepare to spend Rs. 62,000 on the two inputs, what would be the optimum levels of labour, capital and output?

(b) Find the firm's profit. **[Eco. (H) 2006]**

Q6. A monopolist firm has the following demand functions for its product in two markets, denoted by *x* and *y* :

$x = 72 - 0.5P_x$, $y = 120 - P_y$

The combined cost function is $C = x^2 + xy + y^2 + 35$. And the maximum joint production is 40. Thus $x + y = 40$. Find the profit maximizing level of (a) output (b) prices and (c) profit. **[B.B.E. 2004, Eco. (H) 2010]**

Q7. A manufacturer is planning to sell a new product at a price of Rs. 350 per unit and estimates that if *x* thousand rupees is spent on development and *y* thousand rupees is spent on advertisement, consumer will by approximately $\left(\frac{250y}{y+2} + \frac{100x}{x+5}\right)$ units of the product. If manufacturing costs for this product are Rs. 150 per unit, how much should the manufacturer spend on development and how much on advertisement to generate the largest profit.

Q8. A manufacturer has plans to sell a new product at a price of Rs. 150 per unit. He estimates that if he spends *x* thousand rupees on development and *y* thousand rupees on promotion, approximately $\frac{320y}{y+2} + \frac{160x}{x+2}$ units of the product will be sold. The cost of production is Rs. 50 per unit. If the manufacturer has a total of Rs. 8,000 to spend on development and promotion, how should this money be allocated to generate the largest possible profit?

Maximising Utility Subject to Budget Constraint

Q9. Given the utility function $U = (x + 2)(y + 1)$, and the budget constraint $2x + 5y = 51$ find the optimum levels of *x* and *y* purchased by the consumer.

Q10. A consumer's utility function is given as :

$u = \sqrt{x_1 x_2}$

where x_1, x_2 denote the quantities of two products consumed by the consumer and the prices per unit of the goods are Rs. 20 and Rs. 10 respectively. Determine the optimum level of commodities to maximize his utility and spend his total income of Rs. 640 on the two goods.

Q11. If $u = \sqrt{xy}$ is the utility function, $P_x = P_y =$ Rs. 10 and Income = Rs. 100, find the demands that maximize utility. Check the second order condition. **[Eco. (H) 2002, B.B.E. 2005]**

Q12. An individual has a utility function $u = q_1 q_2$ based on the two commodities purchased q_1 and q_2. The prices for these are p_1 and p_2 respectively, while his total income is M. Construct an optimization problem by using the Lagrange method. How much of each commodity should he purchase in order to attain maximum utility? **[Eco. (H) II Sem. 2015(ER)]**

Q13. A consumer has utility function $U(x, y) = (x-a)^\alpha (y-b)^\beta$, where x and y are the quantities of the two goods that he can consume. He buys the goods at fixed prices p_x and p_y, subject to a ceiling M on total expenditure. If he maximizes U, obtain the demand functions for each good in terms of income M and prices p_x and p_y. show that the total expenditure on each good is a linear function of M. **[Eco. (H) 1998]**

Q14. Show that two utility maximizing consumers with utility functions :

$U(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ and $V(x_1, x_2) = x_1^2 x_2^2$

respectively have same demand functions **[Eco. (H) 2008]**

Q15. An individual purchases quantities x_1 and x_2 of two goods whose prices are p_1 and p_2 respectively. His utility function is :

$U(x_1, x_2) = x_1 + \log x_2$.

Assuming his income is M, find the optimal quantities x_1 and x_2. Also find the marginal utility of income. **[Eco. (H) 2007, B.B.E. 2010]**

Q16. A consumer spends an amount m to buy x units of one good at the price 6/unit and y units of a different good at price 10/unit. m is positive. The consumer utility function is

$U(x, y) = xy + y^2 + 2x + 2y$

Find the optimal quantities of x^* and y^* as function of m. what are the solutions for x^* and y^* if $m \le 8$? **[Eco. (H) 2010]**

Q17. A consumer's utility is a function of two goods X and Y and is given by :

U = 100logX + 50logY

If the consumer's budget is Rs. 10 and the price of X is Rs. 3 and the price of Y is Rs. 1, find the quantities of X and Y that the consumer should purchase to maximize utility. What is his marginal utility of Money? **[Eco. (H) 2011]**

Q18. Given the utility function $U = x_1 x_2$ and Budget constraint $y_0 - p_1 x_1 + p_2 x_2 = 0$. Derive the Marshallian demand functions for the goods. **[BBE II Sem. 2013]**

Q19. Let $U(x, y)$ be defined for all $x > 0$, $y > 0$ by $U(x, y) = \ln(x^\alpha + y^\alpha) - \ln y^\alpha$ (α is a positive constant). Is $U(x, y)$ a homogeneous function? Suppose $U(x, y)$ is a utility function for a society with x denoting the economic activity level and y the level of pollution. Assume that the level of pollution y depends on the activity level by the equation $y^3 - ax^4 - b = 0$. (a and b are positive). Find the activity level that maximizes U(x, y) subject to the constraint. **[Eco. (H) 2009]**

Q20. An individual purchases quantities x, y, z of three different commodities whose prices are p, r, s respectively. The consumers income is M where M > 2p and consumers utility function is given by : $U(x, y, z) = x + ln(yz)$.
Find the consumers demand for each good as a function of price and income. Show that the expenditure on good y and z is always equal to the price of the first good.
[Eco. (H) 2009]

Q21. The incomes of an individual in current and next year are Rs. 500 and Rs. 792 respectively. His utility function of the two income streams is $U = x^{1/2}y^{1/2}$, where x and y denote the current and the next year income respectively. If the market rate of interest is 10% p.a., determine the amount the consumer should borrow or lend in current year to maximize his utility. **[Eco. (H) 2001]**

Q22. Consider an individual who maximizes utility function $U(x, y) = \frac{x^{\delta}}{\delta} + \frac{y^{\delta}}{\delta}$, where $\delta \leq 1$, $\delta \neq 0$, subject to the usual linear budget constraint. Show that the constrained utility maximization exercise yields the shares of x and y as :
$S_x = \frac{1}{\left[1 + \left(\frac{p_y}{p_x}\right)^k\right]}$, $S_y = \frac{1}{\left[1 + \left(\frac{p_x}{p_y}\right)^k\right]}$ where k = δ/(δ – 1). If δ tends to zero, what happens to these shares? **[Eco. (H) 2005]**

[**Hint :** Share of x = $\frac{\left(1 - \frac{1}{\eta}\right) MP_x . x}{U}$]

Q23. A consumer's utility function for the two goods is :
$U(x, y) = x^{1/2} y^{1/2}$.
Write down the necessary conditions for the solution of the constrained optimization problem for general values of p_x, p_y and M. Find the optimal values of x and y and the corresponding value of λ. Check the second order conditions. What are the consumer's demand functions for x and y? Find the indirect utility function $U^*(p_x, p_y, M)$ and verify that $\lambda = \frac{\partial U^*}{\partial M}$. **[Eco. (H) II Sem. 2012]**

Q24. Maximize the utility function : **[Eco. (H) II Sem. 2015]**
$$U = f(x, y) = \ln(x) + y$$
Subject to the budget constraint $p_x x + p_y y = M$. Check the second order conditions. What can you comment on the shape of the level curves?

Q25. A consumer's utility function of two goods x and y is given by
$U = \alpha lnx + \beta lny$ (where α, β are positive constants)
Consumer's budget constraint is given by

px + qy = m (where p and q are per unit prices of goods x and y respectively and m is money income).

(i) Using the Lagrangean method, find the optimal values of goods x and y as functions of p, q and m.

(ii) Check the second order condition.

(iii) Find the optimal value function U*(p, q, m). Find $\frac{\partial U^*}{\partial m}$ and give its economic interpretation. **[Eco. (H) II Sem. 2016]**

Q26. A consumer faces the following utility maximization problem
Max $U(x, y) = 100 - e^{-x} - e^{-y}$ subject to px + qy = m, where $x > 0$, $y > 0$. Here, p and q are per unit prices of goods x and y respectively and m is the consumer's money income. **[Eco. (H) II Sem. 2018]**

(i) Find the necessary conditions for the solution of the problem and solve them for the two demand functions x = f(p, q, m) and y = g(p, q, m) by using the Lagrangean method.

(ii) What happens to optimal values of x and y if per unit prices of both goods and consumer's money income are doubled?

Q27. A consumer's utility function for two goods is $U(x, y) = \frac{1}{\beta}\left(x^\beta + y^\beta\right)$, $0 < \beta < 1$. Find the optimal values of x, y or general values of P_x, P_y, M. Verify the second order conditions. Obtain the associated demand functions of x and y. Check if x and y are substitutes. **[Eco. (H) II Sem. 2021]**

Q28. State the necessary & sufficient conditions for the solution of the utility maximization problem, U(x, y) = $U(x, y) = \left(\sqrt{x} + \sqrt{y}\right)^2$ for the general values of p_x, p_y, and M. Find the optimal values of x, y and the corresponding value of λ (the Lagrange multiplier). What are the consumer's demand functions for x and y? Find the indirect utility function $U^*(p_x, p_y, M)$ and verify that $\lambda = \partial U^*/\partial M$.

[Eco. (H) II Sem. 2022]

Answers of Exercise 3

1. L = 10, K = 4,

2.(i) $K = Q.\sqrt{\frac{w}{r}},\ L = Q.\sqrt{\frac{r}{w}}$, **(ii)** same as in (i), **(iii)** C = $2Q\sqrt{rw}$,

3. $L = \frac{X}{A}.\sqrt{\frac{P_K}{P_L}},\ K = \frac{X}{A}\sqrt{\frac{P_L}{P_K}}$, **5.** L = 2000, K = 12,500, Max. Profit = Rs. 76,600,

6. Maximum at $x = 18$, $y = 22$, **7.** $x = 5$, $y = 8$, **8.** $x = 3$, $y = 5$

9. Max. U = 90 at $x = 13$, $y = 5$, **10.** Max. U = 22.63 at $x_1 = 16$, $x_2 = 32$,

11. Max. U = 5 at $x = 5$, $y = 5$, **13.** $x = \dfrac{\frac{\alpha}{\beta}(M - bp_y) + ap_x}{p_x\left(\frac{\alpha}{\beta} + 1\right)},\ y = \dfrac{(M - ap_x) + \frac{\alpha}{\beta}bp_y}{p_y\left(\frac{\alpha}{\beta} + 1\right)}$,

15. $x_1 = \frac{m}{p_1} - 1,\ x_2 = \frac{p_1}{p_2},$ **16.** $x = \frac{40-m}{24}, y = \frac{m-8}{8},$

17. $x = 20/9$, $y = 10/3$ & Marginal utility of money = 15,

19. Homogeneous function of degree 0. For maximum utility $x = y = \left(\frac{b}{1-a}\right)^{1/3}$

20. $x = \frac{M}{p} - 2,\ y = \frac{p}{r},\ z = \frac{p}{s},$

21. $x = 610$, $y = 671$, Consumer will borrow Rs. 110 (610 – 500).

25. (i) $x^* = \frac{m}{p}.\frac{\alpha}{\alpha+\beta}, y^* = \frac{m}{q}.\frac{\beta}{\alpha+\beta}$ (ii) $\frac{\partial U^*}{\partial m} = \frac{\alpha+\beta}{m}$

Basic Concepts

1. **Functions With More Than Two Variables :** For a function Max (or Min)$w = f(x, y, z)$ subject to constraint $g(x, y, z) - k = 0$. The following steps are required :

   (i) Construct an Auxiliary Function (or Lagrange Function) V of defined as : $V(x, y, z, \lambda) = f(x, y, z) - \lambda.[g(x, y, z) - k]$, where λ is known as Lagrange Multiplier.

   (ii) Find the partial derivatives of V w.r.t x, y, z and λ and put them equal to zero, such that $V_x = 0$, $V_y = 0$, $V_z = 0$ and $V_\lambda = 0$ and solve these equations to get the stationary values of x, y and z *eg.*, (x_0, y_0, z_0).

   (iii) Find the partial derivatives $V_{xx}, V_{yy}, V_{zz}, V_{xy}, V_{yz}, V_{xz}$, and g_x, g_y, g_z.

   (iv) Find the value of the Bordered Hessian Determinant

$$|\overline{H}_3| = \begin{vmatrix} 0 & g_x & g_y & g_z \\ g_x & V_{xx} & V_{xy} & V_{xz} \\ g_y & V_{xy} & V_{yy} & V_{yz} \\ g_z & V_{xz} & V_{yz} & V_{zz} \end{vmatrix}$$

   If $|\overline{H}_3| > 0$ then stationary point is maxima and if $|\overline{H}_3| < 0$ then stationary point is minima.

Exercise 4

Q1. Find the possible solution to the problems

$Max\ x^2y^3z$ subject to $x + y + z = 12$

Q2. An individual purchases quantities x, y, z of three different commodities whose prices are p, r, s respectively. The consumers income is M where M > 2p and consumers utility function is given by : $U(x, y, z) = x + ln(yz)$. **[Eco. (H) 2009]** Find the consumers demand for each good as a function of price and income. Show that the expenditure on good y and z is always equal to the price of the first good.

Q3. Consider the problem

$Min\ x^2 + y^2 + z^2$ subject to $x + y + z = 1$.

Using Lagrange Multiplier method find its solution. Also determine if the corresponding maximization problem has any solution.

Q4. Consider the problem

$Min\ x + y + z$ subject to $x^2 + y^2 + z^2 = 1$.

Using Lagrange Multiplier method find its solution.

Q5. Solve the problem

$Min\ x + 4y + 3z$ subject to $x^2 + 2y^2 + \frac{1}{3}z^2 = b$. (Suppose that b > 0 and take it for granted that the problem has a solution.

Answers of Exercise 4

1. $x = 4, y = 6, z = 2,$

2. $x = \frac{M}{p} - 2,\ y = \frac{p}{r},\ z = \frac{p}{s},$

3. $x = 1/3, y = 1/3, z = 1/3,$

4. $x = -\frac{\sqrt{3}}{3}, y = -\frac{\sqrt{3}}{3}, z = -\frac{\sqrt{3}}{3},$

5. The minimum point is (a, 2a, 9a), where a = $-\sqrt{b}/6$,

Basic Concepts

1. **Optimization Under More Than One Constraints by Lagrange's Multiplier Method :** When an objective function Z = $f(x_1, x_2, \ldots x_n)$ is to be optimized subject to a more than one equality constraints

$g_1(x_1, x_2, \ldots x_n) = c_1, g_2(x_1, x_2, \ldots x_n) = c_2, \ldots, g_m(x_1, x_2, \ldots x_n) = c_m$

the following steps are required :

(i) Construct an Auxiliary Function (or Lagrange Function) V defined as :

$V(x_1, x_2, \ldots x_n) = f(x_1, x_2, \ldots x_n) - \sum_{j=1}^{m} \lambda_j.\left(g_j(x1, x2, \ldots\ xn)\right)$.

(ii) Find the partial derivatives of V w.r.t all the variable and put them equal to zero.

(iii) Solve these equation with equality constraints to obtain the values of all variables.

Exercise 5

Q1. Solve the problem

Max(min) $f(x, y, z) = x^2 + y^2 + z^2$

Subject to the constraints

$x + 2y + z = 1$ and $2x - y - 3z = 4$

Q2. Solve the following problem

Min $f(x, y, z) = x^2 - 2x + 2y^2 + z^2 + z$

Subject to the constraints

$x + y + z = 1$ and $2x - y - z = 5$

Q3. By using Lagrange method, find possible solutions to the problem

Max(min) $f(x, y, z) = x + y + z$

Subject to the constraints

$x^2 + y^2 + z^2 = 1$ and $x - y - z = 1$

Q4. Solve the problem

Max(min) $f(x, y, z) = x + y$

Subject to the constraints

$x^2 + 2y^2 + z^2 = 1$ and $x + y + z = 1$

Answers of Exercise 5

1. $f(x,y,z)$ is minimum for $x = \frac{16}{15}, y = \frac{1}{3}, z = -\frac{11}{15}$

2. $f(x,y,z)$ is minimum for $x = 2, y = -\frac{1}{6}, z = -\frac{5}{6}$

3. $f(x,y,z)$ is minimum for $x = -\frac{1}{3}, y = -\frac{2}{3}, z = -\frac{2}{3}$, and $f(x,y,z)$ is maximum for $x = 1, y = 0, z = 0$.

4. $f(x,y,z)$ is minimum for $x = 0, y = 0, z = 1$, and $f(x,y,z)$ is maximum for $x = \frac{4}{5}, y = \frac{2}{5}, z = -\frac{1}{5}$.

Basic Concepts

1. **Economic Interpretations of Lagrange Multipliers :**

When an objective function Z = $f(x_1, x_2, \dots x_n)$ is to be optimized subject to a more than one equality constraints

$g_1(x_1, x_2, \dots x_n) = c_1$, $g_2(x_1, x_2, \dots x_n) = c_2$, …, $g_m(x_1, x_2, \dots x_n) = c_m$

then the Lagrange Multiplier $\lambda_i = \lambda_i(c)$ for the i[th] constraint is the rate at which the optimal value of the objective function changes w.r.t. changes in the constant c_i. The number λ_i is referred to as a shadow price or marginal value imputed to a unit of resource i.

Suppose we change the constants $c = (c_1, c_2, \dots, c_m)$ by $dc = (dc_1, dc_2, \dots, dc_m)$ then

$df^*(c) = \sum_{j=1}^{m} \lambda_j . dc_j$

And, $f^*(c + dc) = f^*(c) + \sum_{j=1}^{m} \lambda_j . dc_j$

Exercise 6

Q1. Consider the problem

Max(min) $f(x,y,z) = x^2 + y^2 + z^2$

Subject to the constraints

$x + 2y + z = 1$ and $2x - y - 3z = 4$

(i) Solve this problem using Lagrange Multiplier Method.

(ii) Suppose we change the first constraint to $x + 2y + z = 0.9$ and the second constraint to $2x - y - 3z = 4.1$, estimate the new value of the objective function.

Q2. Consider the problem Max $f(x,y) = x + y$ Subject to $g(x,y) = x^2 + y = 1$

(i) Write down the Lagrangean function for the problem and solve the necessary conditions in this case.

(ii) If the constraint is replaced by $g(x,y) = x^2 + y = 1.1$ then find the change in the optimal value of the objective function.

Q3.(i) Solve the following problem

Max(min) $f(x,y) = x + y$

Subject to the constraints

$x^2 + 2y^2 + z^2 = 1$ and $x + y + z = 1$.

(ii) Suppose the constraints changes to $x^2 + 2y^2 + z^2 = 0.9$ and $x + y + z = 1.2$

What is the approximate change in the maximum/minimum values of the objective function.

Answers of Exercise 6

1(i). $f(x,y,z)$ is minimum for $x=\frac{16}{15}, y=\frac{1}{3}, z=-\frac{11}{15}$

(ii). $df^*(c)=\frac{0.2}{75}$, New $f^*(c)=\frac{402.6}{225}$

2.(i) $x = ½, y = ¾$, **(ii)** $df^*(c)=0.1$,

3. $f(x,y,z)$ is minimum for $x=0, y=0, z=1$, and $f(x,y,z)$ is maximum for $x=\frac{4}{5}, y=\frac{2}{5}, z=-\frac{1}{5}$.

Basic Concepts

1. **Envelope Theorem :** This theorem is used to determine the change in the value of objective function due to change in the value of a parameter. Let z = $f(x, y, a)$ be an objective function with a parameter a. then first of all find the point at which the function is maximum/minimum. Then find the value $\frac{\partial z}{\partial a}$ at that point, represents the rate of change of optimum value of the objective function as parameter 'a' changes. This optimum value represented as z* can also be obtained by substituting the value of x and y at optimum point in the given function z = $f(x, y, a)$

Exercise 7

Envelope Theorem

Q1. Determine the effect of an increase in the value of a on the optimum value of the function $y=2x^2-ax+a^2$.

Q2. Determine whether the function $z=x^2+y^2-2ax-4ay+a^2$ has a maxima or minima, where a is a positive parameter. If z* denote the optimum value of the function, find $\frac{dz^*}{da}$ by envelope theorem. Verify your answer by alternative method.

Q3. Determine whether the function $z=4a-x^2-y^2$ subject to the constraints ax +3ay = 10 has a maxima or minima, where a is a positive parameter. If z* denote the optimum value of the function, find $\frac{dz^*}{da}$ by envelope theorem. Verify your answer by alternative method.

Q4. A firm uses inputs L and K to produce a target level of output Q = LK. The prices per unit of L and K are w and r, respectively. Using the Lagrangian method solve the following minimization problem : **[Eco. (H) II Sem. 2014]**

$$\underset{L,K}{Min}\ C(L,K)=wL+rK$$

Subject to

Q = LK

(i) Find the cost minimizing inputs L* and K*.
(ii) Find the optimal value function C* as a function of w, r and Q.
(iii) Apply the envelope theorem to find the derivative of the optimal value function C*(w, r, Q) with respect to Q.

Q5. $x(a,b) = A\sqrt{ab}$ is a production function for good x and y using inputs a and b. Use the Lagrangian method to find the amount of factors required to produce an output $\bar{x}$ at minimum cost, when prices of inputs are p_a and p_b. Check the second order conditions and verify the envelope theorem.

Q6. Let $f(x,y) = (x^2 - \alpha xy)e^y$ be a function of two variables with α as a constant and α ≠ 0.
(i) Find the critical points of f and decide for each of them if it is a local maximum point, local minimum point or a saddle point.
(ii) Let (x^*, y^*) be the critical point where $x^* \neq 0$ and let $f^*(\alpha) = f(x^*, y^*)$. Find $\frac{df^*(\alpha)}{d\alpha}$ and show that if we define $\hat{f}(x,y,\alpha) = (x^2 - \alpha xy)e^y$ then $\frac{\partial f(x^*, y^*, \alpha)}{\partial \alpha} = \frac{df^*(\alpha)}{d\alpha}$. **[Eco. (H) II Sem. 2022]**

Answers of Exercise 7

1. $\frac{dy^*}{da} = \frac{\partial y}{\partial a} = \frac{7a}{4}$, **2.** Minima, $\frac{dz^*}{da} = -4a$, **3.** Maxima, $\frac{dz^*}{da} = -4$