Chapter 2

The Solow Model

Learning Objectives :

After learning this chapter you will understand :

- **Solow Growth Model.**
- **The Steady State Level of Capital.**
- **Effects of Changes in Saving Rate.**
- **Golden Rule Level of Capital.**
- **Transition to the Golden Rule Level of Capital.**
- **Effects of Population Growth in the Solow Model.**
- **Technological Progress in the Solow Model.**
- **Total Factor Productivity.**

Basic Concepts

1. **Solow Growth Model :** The Solow growth model is named after economist Robert Solow. Solow Growth Model explains that why the economics of some countries gross faster than other countries. The Solow model is designed to show how growth in capital stock, growth in labour force and advances in technology interact in an economy to affect the level of an economies output and its growth overtime.
2. **Assumptions :** The Solow model is based upon the following assumptions :
   (i) The world will consist of countries that produce and consume only a single homogeneous good, we call it output.
   (ii) Technology is exogeneous, *i.e.*, the technology available to firms in this simple world is unaffected by the actions of the firms, including research and development.
   (iii) Labour is an exogenous variable which grows at some constant rate of *n* per year.
   (iv) The production function exhibits constant returns to scale, *i.e.*, if all the inputs are doubled the level of output also gets doubled.
   (v) The factor market is perfectly competitive and the firms pays worker a wage *w* for each unit of labour and rent *r* for each unit of capital.
   (vi) The economy is closed, savings equal investment and the only use of investment in this economy is to accumulate capital.
3. **The Basic Solow Model :** The Solow model is built around two equations, a production function and a capital accumulation equation.

   The production function : The production function describes how inputs such as bulldozers, semiconductors, engineers, and steel-workers combine to produce output.1 To simplify the model, we group these inputs into two categories, capital, K, and labor, L, and denote output as Y. The production function is assumed to have the Cobb-Douglas form and is given by

   $$Y = F(K, L) = K^{\alpha}L^{1-\alpha}$$

   where α is some number between 0 and 1. This production function exhibits constant returns to scale, *i.e.*, if all of the inputs are doubled, output will exactly double. Solow model assumes that production function exhibits constant returns to scale, *i.e.*, if labour and capital both are multiplied by a constant 'λ' then output also gets multiplied by the same constant 'λ' because production function gives constant returns to scale, *i.e.*,

   $$Y = F(K, L) = K^{\alpha}L^{1-\alpha}$$

   So, $F(\lambda K, \lambda L) = (\lambda K)^{\alpha}(\lambda L)^{1-\alpha} = (\lambda)^{\alpha+1-\alpha}(K)^{\alpha}(L)^{1-\alpha}$

   $= \lambda F(K, L)$

   Since, we express things in terms of per worker, so we can also express the production function in per worker terms as under :

   $$Y = K^{\alpha}L^{1-\alpha}$$

   On dividing both sides by L we get

   $$\frac{Y}{L} = \frac{K^{\alpha}L^{1-\alpha}}{L} = \left(\frac{K}{L}\right)^{\alpha} \Rightarrow \quad y = k^{\alpha}$$

Where, y = Y/L, *i.e.*, Output per worker and
k = K/L, *i.e.*, Capital per worker.

Because we have assumed that production function gives constant returns to scale so the size of economy, as measured by number of workers, does not effect the relationship between output per worker and capital per worker.

Slope of Production Function : The slope of production function shows how much extra output a worker produces when given an extra unit of capital. So the slope of production function is marginal product capital, *i.e.*, if k increases by 1 unit then output increases by MP_k units.

$$\text{Slope} = MP_K = f(k + 1) - f(k) = \partial f/\partial k$$

The production function becomes flatter as k increases indicating that it gives diminishing marginal product of capital. When k is low so the average worker has little capital to work with so an additional unit of capital is very useful and produces a lot of additional output. But when capital is very high then average worker has a lot of capital so additional unit produces very less output.

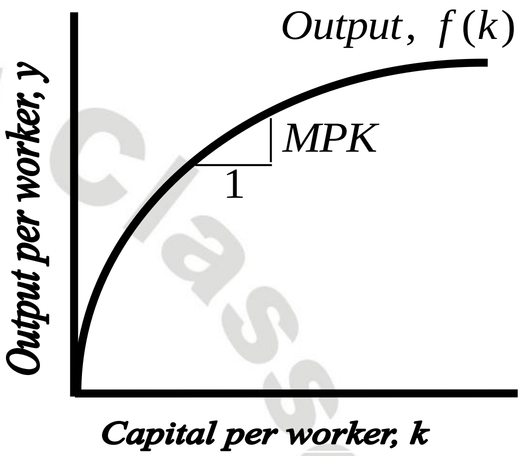


Figure 1 : The Production Function

Diminishing Marginal Productivity of Capital : With more capital per worker, firms produce more output per worker. However, there are diminishing returns to capital per worker: each additional unit of capital we give to a single worker increases the output of that worker by less and less.

Factor Payments and Profit Maximization : Firms in this economy pay workers a wage, w, for each unit of labor and pay r in order to rent a unit of capital for one period. Normalizing the price of output in our economy to unity, profit-maximizing firms solve the following problem :

$$\text{Max } \pi(K, L) = [F(K, L) - rK - wL]$$

According to the first-order conditions for this problem, firms will hire labor until the marginal product of labor is equal to the wage and will rent capital until the marginal product of capital is equal to the rental price, thus

$$w = \partial F/\partial L = (1 - \alpha)Y/L \text{ and } r = \partial F/\partial K = \alpha\, Y/K$$

So the share of output paid to labour = $wL/Y = 1 - \alpha$ and share of output paid to capital = $rK/Y = \alpha$, both are constants.

The Capital Accumulation Equation : The second key equation of the Solow model is an equation that describes how capital accumulates. The capital accumulation equation is given by

$$\dot{K} = sY - \delta K$$

According to this equation, the change in the capital stock, $\dot{K}$ is equal to the amount of gross investment, sY, less the amount of depreciation that occurs during the production process, δK.

The term sY of above equation represents gross investment. Following Solow, we assume that workers/consumers save a constant fraction, s, of their combined wage and rental income, Y = wL + rK. The economy is closed, so that saving equals investment, and the only use of investment in this economy is to accumulate capital. The consumers then rent this capital to firms for use in production

The term δK of above equation reflects the depreciation of the capital stock that occurs during production. The standard functional form used here implies that a constant fraction, δ, of the capital stock depreciates every period (regardless of how much output is produced).

The Capital Accumulation Equation in per worker terms : The capital accumulation equation in per worker terms can be expressed as under :

Since capital per worker (k) = K/L

Taking log , we get

$$\log k = \log K - \log L$$

on differentiating with respect to t, we get

$$\dot{k}/k = \dot{K}/K - \dot{L}/L$$

In the Solow model it is assumed that growth rate of labour is constant which is n, *i.e.*, $\dot{L}/L = n$, therefore

$$\dot{k}/k = \dot{K}/K - n$$

We know that

$$\dot{K} = sY - \delta K \quad \Rightarrow \quad \dot{K}/K = sY/K - \delta$$

Therefore,

$$\dot{k}/k = sY/K - \delta - n$$

$$\Rightarrow \quad \dot{k}/k = (s(Y/L))/((K/L)) - \delta - n$$

$$\Rightarrow \quad \dot{k}/k = sy/k - (\delta + n)$$

This now yields the capital accumulation equation in per worker terms :

$$\dot{k} = sy - (\delta + n)k$$

This equation says that the change in capital per worker each period is determined by three terms. Two of the terms are analogous to the original capital accumulation equation. Investment per worker, sy, increases k, while depreciation per worker, δk, reduces k. The term that is new in this equation is a reduction in k because of population growth, the nk term. Each period, there are nL new workers around who were not there during the last period. If there were no new investment and no depreciation, capital per worker would decline because of the increase in the labor force.

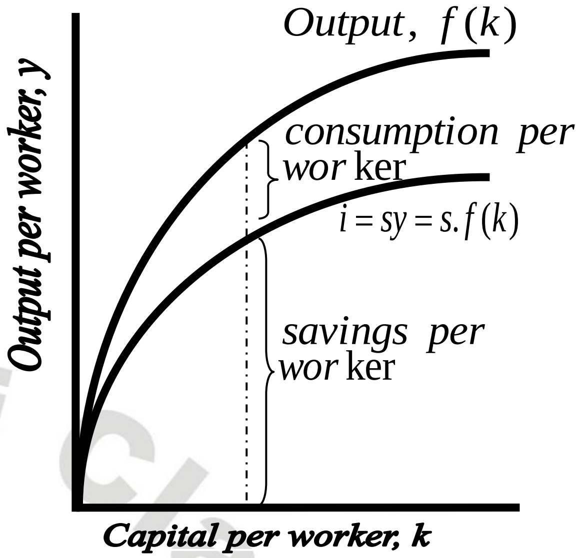


Figure 2 : Consumption and Savings

Saving : In the Solow Growth Model, we assume that people save a fraction of their income and consume a fraction of their income every year. Let '*s*' represent the fraction of saving to the fraction of consumption is 1 – *s*.

Savings = $s.y$
Consumption = $(1 - s)y$
Now,

$y = c + i$

$\Rightarrow \quad y = (1 - s)y + i$

$\Rightarrow \quad y = y - s.y + i$

$\Rightarrow \quad i = s.y = s.f(k) \qquad [\because y = f(k)]$

$\Rightarrow$ investment per worker = saving per worker

This equation shows that investment equals saving. Thus, the rate of saving '*s*' is also fraction of output devoted to investment.

4. **Solving the Solow Model :** The Solow diagram consists of two curves, plotted as functions of the capital-labor ratio, *k* (= K/L). The first curve is the amount of investment per person, $sy = sk^{\alpha}$. This curve has the same shape as the production function plotted in Figure 2, but it is translated down by the factor s. The second curve is the line $(n + \delta)k$, which represents the amount of new investment per person required to keep the amount of capital per worker constant—both depreciation and the growing workforce tend to reduce the amount of capital per person in the economy. The difference between these two curves is the change in the amount of capital per worker.

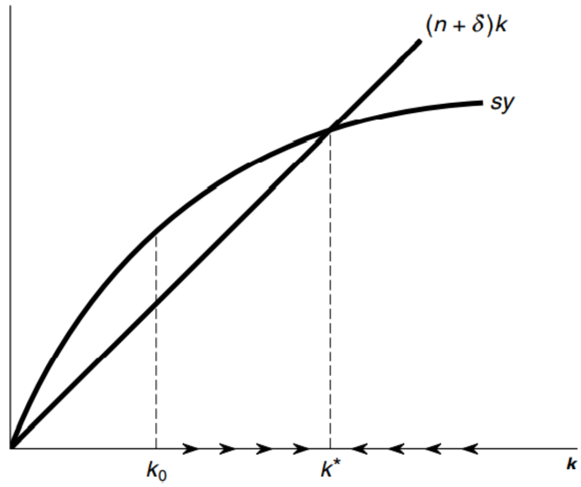


Figure 3 : Steady State Level

Steady State Level of Capital : The steady state level of capital is that level at which the capital stock will not change because the two forces which change the capital are equal. Investment causes an increase in capital stock whereas depreciation and population growth reduces it, at steady state level of capital we have $\Delta k = 0$, *i.e.*, $i = (n + \delta)k$. In ***Figure 3***, k^* represents the steady state level of capital because at k^* we have $i = (n + \delta)k$.

The steady state level of capital k^* is the level at which investment equals $(n + \delta)k$, indicating that the amount of capital will not change over time, i.e., $\Delta k = 0$. Below k^*, investment exceeds $(n + \delta)k$, so the capital stock grows and above k^*, $(n + \delta)k$ exceeds investment so the capital stock shrinks. The steady state level of capital is important for two reasons :

(i) The economy which is at steady state level will remain at steady state.

(ii) The economy which is not at steady state level will move towards steady state in the long run so steady state level is also known as long run equilibrium of the economy.

The steady state level of capital is also known as long run equilibrium. ***Figure 4***, illustrates the steady state level of capital. Here, k^* is the steady state level of capital, y^* is the steady state level of output per worker, c^* is the steady state level of consumption per worker and s* is the steady state level of saving per worker.

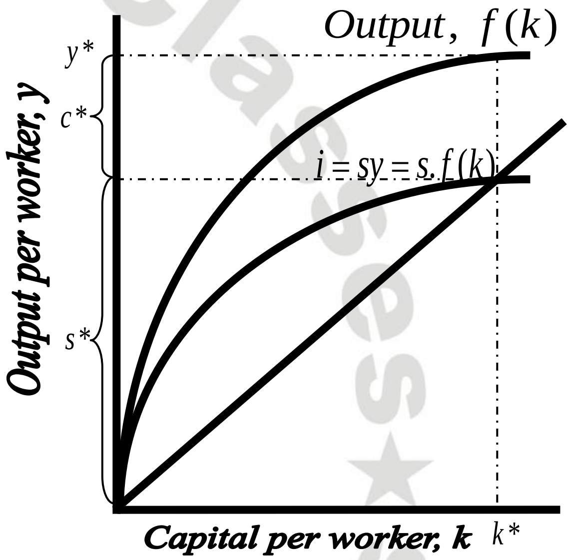


Figure 4 : Steady State Level of Capital

Capital Deepening : When investment exceeds $(n + \delta)k$, as at k_0 in ***Figure 3***, the change in capital stock per worker is positive and the economy is increasing its capital per worker, we say that ***capital deepening*** is occurring.

Capital Widening : When investment equals $(n + \delta)k$, the change in capital stock per worker is zero but the actual capital stock K is growing (because of population growth), we say that only ***capital widening*** is occurring. This means even at steady state level where capital per worker (k) is constant, the total capital stock is growing at the rate of population growth, *i.e.*, n and we call it ***capital widening***.

5. **Mathematical Solution of Solow Model :** By definition, the steady-state quantity of capital per worker is determined by the condition that $\dot{k} = 0$. Since the capital accumulation equation is $\dot{k} = sy - (\delta + n)k$, where $y = k^{\alpha}$ and at steady state level capital per worker is represented as k*, therefore we have,

$$\dot{k} = 0$$

$$\Rightarrow \quad s[(k^*)]^{\alpha} = (\delta + n)k^* \quad \Rightarrow \quad \frac{s}{\delta + n} = [(k^*)]^{1-\alpha}$$

$$\Rightarrow \quad k^* = [\frac{s}{\delta + n}]^{1/(1-\alpha)}$$

Substituting this into the production function reveals the steady-state quantity of output per worker, y^* :

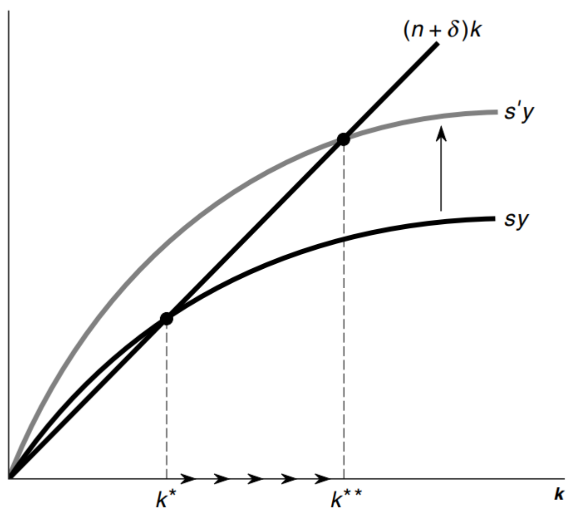
$$y^* = [\frac{s}{\delta+n}]^{\alpha/(1-\alpha)}$$

Notice that the endogenous variable y^* is now written in terms of the parameters of the model. Thus, we have a "solution" for the model, at least in the steady state.

This equation reveals the Solow model's answer to the question "Why are we so rich and they so poor?" Countries that have high savings/investment rates will tend to be richer, ceteris paribus. Such countries accumulate more capital per worker, and countries with more capital per worker have more output per worker. Countries that have high population growth rates, in contrast, will tend to be poorer, according to the Solow model. A higher fraction of savings in these economies must go simply to keep the capital-labor ratio constant in the face of a growing population. This capital-widening requirement makes capital deepening more difficult, and these economies tend to accumulate less capital per worker

Note **There are an infinite number of possible steady states, some higher than others. Which steady state our economy is in (and therefore what output we have) depends on where the *s.f(y)* curve meets the $(\delta + n)k$ curve, which in turn depends on the savings rate *s* in the economy.**

6. **Effects of Change in Saving Rate :** Consider an economy that has arrived at its steady-state value of output per worker. Now suppose that the consumers in that economy decide to increase the investment rate permanently from s to some value $s^{/}$. The increase in the investment rate shifts the s*y* curve upward to $s^{/}y$. At the current value of the capital stock, k^*, investment per worker now exceeds the amount required to keep capital per worker constant, and therefore the economy begins capital deepening again, i.e., the capital stock will continue to rise until new steady state k^{**} is obtained . This capital deepening continues until $s^{/}y = (n + \delta)k$ and the capital stock per worker reaches a higher value, indicated by the point k^{**}. From the production function, we know that this higher level of capital per worker will be associated with higher per capita output; the economy is now richer than it was before.



The Solow model shows that the saving rate is a key determinant of the steady state capital. If the saving rate is high, the economy will have a large capital stock and a high level of output in the steady state. Whereas If the saving rate is low, the economy will have a small capital stock and a low level of output in the steady state.

7. **Relation Between Saving Rate and Economic Growth :** According to Solow model, higher savings leads to faster growth but only temporarily. An increase in the rate of saving raises growth only until the economy reaches the new steady state. The higher rate of savings yield higher capital stock and higher output levels at the steady state level but it will not maintain a higher rate of growth forever. A higher saving rate is said to have a *level effect*, because only the level of income per person is influenced by the saving rate in the steady state.

8. **Relation between Saving and Investment :** The data collected from various economies shows that there is direct relationship between the fraction of output devoted to investment and the level of income per person, *i.e.*, the economies with higher saving rate have high levels of income whereas the economies with low saving rate will have low level of income. But the relationship between these two variables are far from perfect. There are many economies in the world which have same level of saving but different level of income. There are various reasons for this like political situation, cultural differences, stability of financial market etc.

Exercise 1

Theory Questions

Q1. In the Solow model, how does the saving rate affect the steady state level of income?

Q2. In the Solow model, how does the saving rate affect the steady state rate of growth?

Q3. Consider the following statement : "Devoting a large share of national output to investment would help to restore rapid productivity growth and increasing living standards." Do you agree with this statement in the context of the Solow Model. Explain? **[Eco. (H) 2016]**

Q4. Explain why an increase in saving in the Solow model has a level effect but not a growth effect. **[Eco. (H) 2015]**

Numerical Problems

Q1. Consider an economy with the production function $Y = K^{\alpha} L^{1-\alpha}$. Assume that α = 1/3. **[Eco. (H) 2017]**

(i) Is this production function characterized by Constant Returns to Scale?

(ii) In terms of saving rate s and depreciation rate δ, derive expressions for capital per worker, output per worker and consumption per worker in the steady state.

(iii) Suppose $\delta = 0.08$, and $s = 0.32$ what is the steady state capital per worker, output per worker and consumption per worker?

[**Ans : (i)** Yes, **(ii)** $k^* = \left[\frac{s}{\delta}\right]^{3/2}, y^* = \left[\frac{s}{\delta}\right]^{1/2} \; and \; c^* = (1-s)\left[\frac{s}{\delta}\right]^{1/2}$,
(iii) $k^* = 8, y^* = 2, c^* = 1.36$**]**

Basic Concepts

1. **Population Growth in the Solow Growth Model :** Instead of assuming that the population is fixed, we now suppose that the population and the labour force grow at a constant rate *n*. Capital per worker is k = K/L and output per worker is y = Y/L. If labour force (L) increases then it with decrease capital per worker and also output per worker so that increase in labor force causes a decreases in capital. The change in capital per worker can be represented as $\Delta k = i - (\delta + n)k$. Here '*n*' represent the rate at which labor force is growing. This equation shows that investment, depreciation and population growth changes the capital per worker, investment increases the capital per worker whereas depreciation and population growth decreases capital per worker.

   Depreciation causes decrease in capital due to wear and tear. Whereas increase in labour force decreases capital per worker by spreading it more thinly among the population.
2. **Break Even Level of Investment :** The level of investment which is necessary to keep the capital per worker constant is called break even level of investment. Depreciation and population growth are two reasons due to which the capital per worker shrinks. If *n* is the rate of population growth and δ is the rate of depreciation then $(\delta + n)k$ represents the break even level of investment, *i.e.*, the amount of investment necessary to keep the capital per worker *k* as constant.
3. **Steady State with Population Growth :** An economy is in steady state if the capital per worker is not changing. Let k^* denotes the steady state level of capital so at k^* investment is equal to depreciation and population growth, *i.e.*, $i = (\delta + n)k$

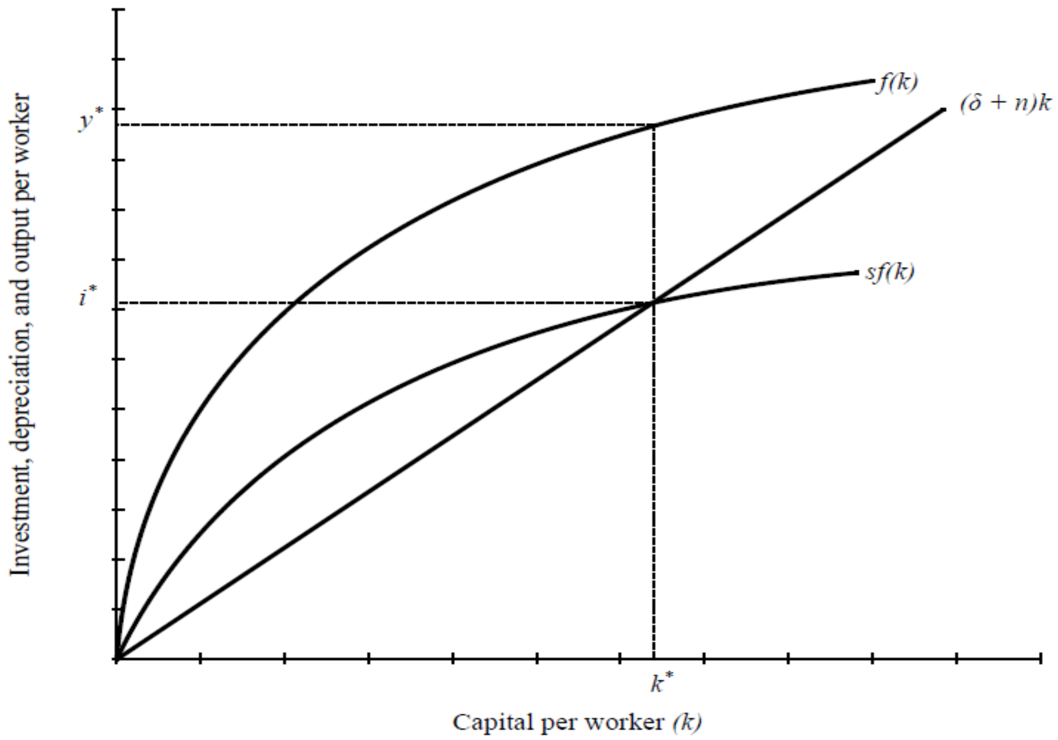


Figure 1 : Steady State Level with Depreciation and Population Growth

As shown in ***Figure 1***, in order to retain an unchanging level of capital per worker k over time, we have to invest enough to create new capital to offset this loss over time. Thus, to maintain a "steady state" where capital per worker is constant over time, we must have that :

$$\Delta k = s.f(k) - (\delta + n)k = 0$$

$$\Rightarrow \quad s.f(k^*) = (\delta + n)k^*$$

where * indicates steady state values. Note how this shows that, as our capital per worker k gets larger, larger amounts of investment are required to maintain $\Delta k = 0$. The economy will always work itself to a steady state point. If the rate of capital replenishment is greater than the loss due to depreciation and population growth, *i.e.*, $s.f(k) > (\delta + n)k$, then the capital stock will grow. If the rate of replenishment is lower than depreciation plus population growth, *i.e.*, $s.f(k) < (\delta + n)k$, then the capital stock will shrink. Only when the two are equal will there be no further adjustment to the capital stock in the economy.

Deriving Steady State Consumption

We know that

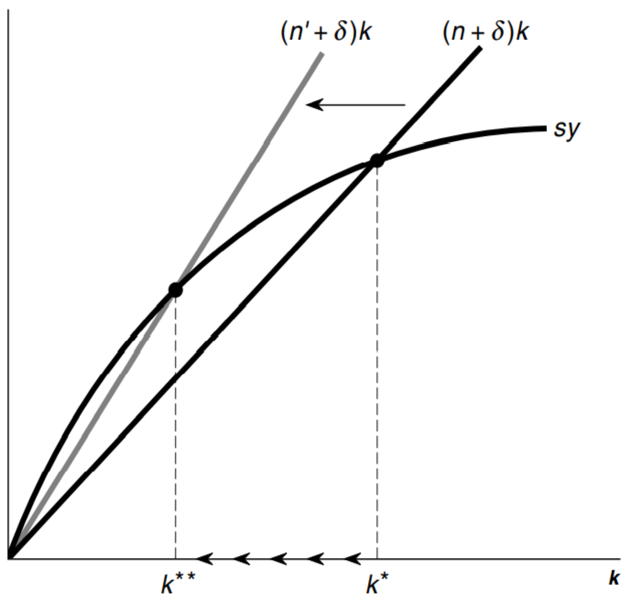
$$y = c + i \quad \Rightarrow \quad c = y - i \quad \Rightarrow \quad c = f(k) - i$$

at steady state level of capital, c^* is consumption, $f(k^*)$ is output, and $i = (\delta + n)k^*$ is break even investment.

$$\therefore \quad c^* = f(k^*) - (\delta + n)k^*$$

Note **There are an infinite number of possible steady states, some higher than others. Which steady state our economy is in (and therefore what output we have) depends on where the $s.f(y)$ curve meets the $(\delta + n)k$ curve, which in turn depends on the savings rate s in the economy.**

4. **Effects of Change in Population Growth Rate :** Suppose an economy has reached its steady state, but then because of immigration, for example, the population growth rate of the economy rises from n to n'. Due to this increase in population growth rate the $(n + \delta)k$ curve rotates up and to the left to the new curve $(n' + \delta)k$. At the current value of the capital stock, k^*, investment per worker is now no longer high enough to keep the capital-labor ratio constant in the face of the rising population. Therefore the capital-labor ratio begins to fall. It continues to fall until the point at which $sy = (n' + \delta)k$, indicated by k^{**} in



adjoining Figure. At this point, the economy has less capital per worker than it began with and is therefore poorer: per capita output is ultimately lower after the increase in population growth in this example.

5. **Effects of Population Growth :** The population growth has the following effects in the Solow Model
   (i) Population growth explains sustained growth in total output. At the steady state level of capital the population growth, the capital per worker and the output per worker all are constant. Since number of workers are growing at constant rate n, however, total capital and total output must also be growing at rate *n*.
   (ii) If the population growth rate increases then steady state level declines. Thus, the Solow model predicts that economies with higher rates of population growth will have lower levels of capital per worker and therefore lower incomes.

Note **A change in population growth rate, like a change in the saving rate, has a level effect on the income per person, but does not affect the steady state growth rate of income per person.**

Exercise 2

Theory Questions

Q1. Discuss the steady state condition in the Solow model with population growth and no technological progress. **[Eco. (H) 2011]**

Q2. In the Solow model, how does the rate of population growth affect the steady state level of income?

Q3. Show that in the steady state of the Solow model without technological progress, the capital per worker and output per worker are constant but total capital and total output grow at the rate of population growth.

Q4. Assuming zero growth rate of technology show the effect of an increase in the stock of labour and decrease in investment rate on the economy's level of output in the context of the Solow model. **[Eco. (H) 2007]**

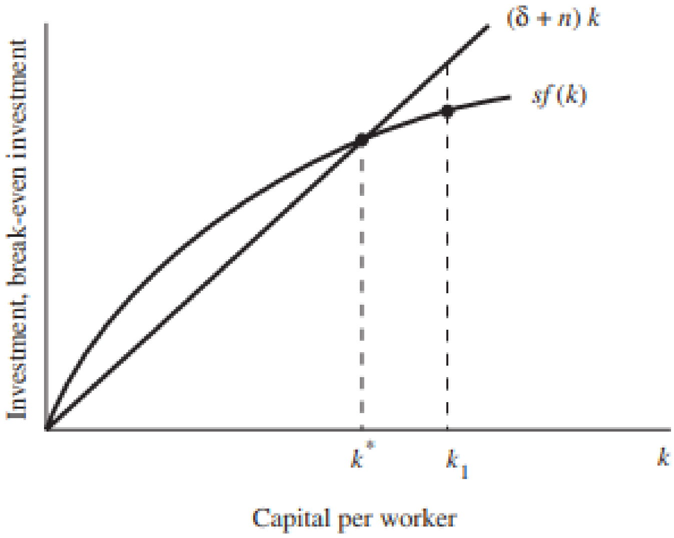
Q5. Consider an economy characterized by the Solow Model this is initially in steady state. A war reduces the capital stock in this economy while leaving the labour unchanged. The war is accompanied by a decline in the saving rate of the economy. Illustrate the immediate and the long term impact of the war and the change in the saving rate on the capital per capita in the economy. **[Eco. (H) 2016]**

Q6. Consider an economy with a fixed saving rate and no technological progress. Suppose there is a war that does not directly affect the capital stock but the casualties reduce the labour force. What is the immediate impact on total output and on output per worker? Assuming that the saving rate is unchanged and that the economy was in a steady state before the war, what happens subsequently to output per worker in a post war economy? Does the growth rate of output per worker increase or decrease after the war? **[Eco. (H) 2017, 2018]**

Answer

(i) The production function in the Solow growth model is Y = F(K, L), or expressed terms of output per worker, $y = f(k)$. If a war reduces the labor force through casualties, then L falls but k = K/L rises. The production function tells us that total output falls because there are fewer workers. Output per worker increases, however, since each worker has more capital.

(ii) The reduction in the labor force means that the capital stock per worker is higher after the war. Therefore, if the economy were in a steady state prior to the war, then after the war the economy has a capital stock that is higher than the steady state level. This is shown in adjoining Figure, as an increase in capital per worker from k^* to k_1. As the economy returns to the steady state, the capital stock per worker falls from k_1 back to k^*, so output per worker also falls.



(iii) There is no per capita growth in this version of the model! Output per worker (and therefore per person, since we've assumed the labor force participation rate is constant) is constant in the steady state. Output itself, Y, is growing, of course, but only at the rate of population growth. The reduction in the labor force means that the output per worker is higher after the war. Therefore, if the economy were in a steady state prior to the war, then after the war the economy has output that is higher than the steady state level. So, there will negative growth in output per worker until economy returns to the steady state.

Q7. Suppose in the Solow model output in a country is produced with the production function $Y = AK^{\alpha}L^{1-\alpha}$, where $0 < \alpha < 1$. Capital depreciates at the rate δ and productivity, A, and saving rate s, is assumed constant. Suppose initially economy is at steady state with constant population growth rate n_1. Depict the steady state equilibrium in a graph. Now suppose there is a one time increase in immigration, perhaps due to the influx of refugees from a troubled foreign country. Use the previous graph to illustrate the short run and long run effects of this on domestic economy. **[Eco. (H) 2019]**

Q8. Examine in the context of the Solow model the short and long run effects of a one time permanent increase in the stock of labour. Assume that g (growth of technology) = 0 and n (population growth rate) > 0. **[Eco. (H) 2022]**

Basic Concepts

1. **Economic Growth in the Simple Solow Model :** There is no per capita growth in this version of the model! Output per worker (and therefore per person, since we've assumed the labor force participation rate is constant) is constant in the steady state. Output itself, Y, is growing, of course, but only at the rate of population growth. This version of the model generates differences in per capita income across countries. It generates a constant capital-output ratio (because both *k* and *y* are constant, implying that K/Y is constant). It generates a constant interest rate, the marginal product of capital. However, it fails to predict a very important stylized fact: that economies exhibit sustained per capita income growth. In this model, economies may grow for a while, but not forever. For example, an economy that begins with a stock of capital per worker below its steady-state value will experience growth in *k* and *y* along the transition path to the steady state. Over time, however, growth slows down as the economy approaches its steady state, and eventually growth stops altogether, when economy reaches its steady state.

To see that growth slows down along the transition path, notice two things.

First, from the capital accumulation equation $\dot{k} = sy - (\delta + n)k$

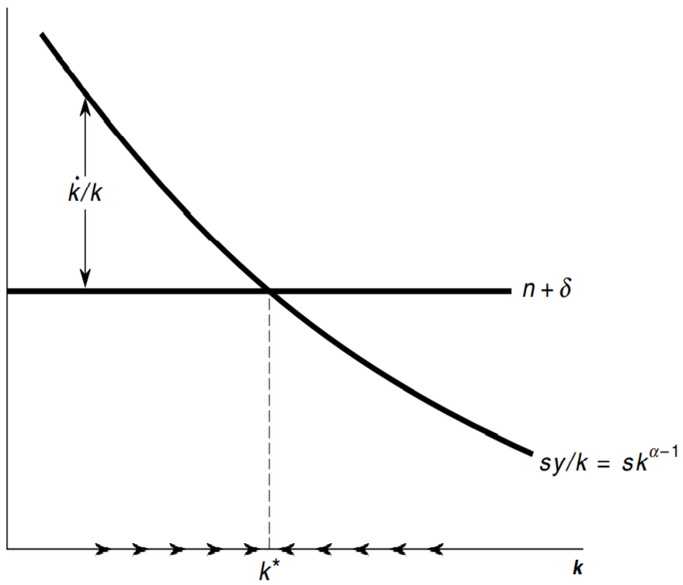
If we divide both sides by *k* to get

$$\frac{\dot{k}}{k} = \frac{sy}{k} - (\delta + n) \quad \Rightarrow \quad \frac{\dot{k}}{k} = \frac{sk^{\alpha}}{k} - (\delta + n)$$

$$\Rightarrow \quad \frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + n) \qquad \text{(i)}$$

Because α is less than one, as *k* rises, the growth rate of *k* gradually declines.

Second, we know that $y = k^{\alpha} \quad \Rightarrow \quad \dot{y} = \alpha k^{\alpha-1}\dot{k} \quad \Rightarrow \quad \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$, *i.e.*, the growth rate of *y* is proportional to the growth rate of *k*, so that the same statement holds true for output per worker.



The transition dynamics implied by equation (i) are plotted in the adjoining Figure. The first term on the right-hand side of the equation is $sk^{\alpha-1}$, which is equal to sy/k. The higher the level of capital per worker, the lower the average product of capital, y/k, because of diminishing returns to capital accumulation (α is less than one). Therefore, this curve slopes downward. The second term on the right-hand side of equation (i) is $n + \delta$, which doesn't depend on *k*, so it is plotted as a horizontal line. The difference between the two lines in the adjoining figure is the growth rate of the capital stock, or $\frac{\dot{k}}{k}$. Thus, the figure clearly indicates that the further an economy is below its steady-state value of k, the faster the economy grows. Also, the further an economy is above its steady-state value of k, the faster k declines.

Basic Concepts

1. **Golden Rule Level of Capital :** The higher rate of saving yields greater income but the objective of the economist is to maximize the economic well being. For the sake of simplicity we assume that the policy maker can set the saving rate at any level.

In the Solow Growth Model, a steady state saving rate of 100% implies that all the income is going to investment implying a steady state consumption level of 0%. But if the saving rate is 0%, *i.e.*, no investment is being made so the capital stock depreciate without replacement, this makes steady stage unsustainable except at zero output which again implies consumption level of zero. Thus, somewhere in between is the level of savings where the level of consumption is at its maximum possible value, the corresponding steady state level of capital is called Golden Rule level of Capital and it is denoted by k^*_{gold} .

When choosing a steady state, the policymaker's goal is to maximize the well being of the individuals who make up the society. Individuals themselves do not care about the amount of capital in the economy or even the amount of output. They care about the amount of goods and services they can consume. Thus, a benevolent policymaker would want to choose the steady state with the highest level of consumption.

2. **How to Determine if the Economy is at Golden Rule Level of Capital :** The following steps are used to determine if the economy is at Golden Rule Level of capital.

(i) We know that income is the sum of consumption and investment so income per worker is the sum of consumption per worker and investment per worker, *i.e.*, $y = c + i \Rightarrow c = y - i$, *i.e.*, consumption per worker is simply output per worker minus income per worker.

(ii) At steady level of capital k^*, investment is equal to depreciation. So $i = \delta k^*$ and steady state output is $f(k^*)$. So replacing the value of output and investment we get

$$c^* = f(k^*) - \delta k^*$$

Thus, steady state consumption is steady state output minus state depreciation. This equation implies that there are two opposing effects of increase in steady state capital on steady state consumption :

(i) An increase in steady state capital increases output.

(ii) An increase in steady state capital means more depreciation.

The higher level of capital effects both output and depreciation. If the capital stock is below the golden rule level then an increase in capital stock raises the output more than the depreciation so the consumption increases. Whereas when capital stock is above the golden rule level then increase in capital increases production but increase in output is less than depreciation so consumptions falls. Thus, there is a point between these two level where increase in output is equal to depreciation. So, at this point consumption is maximum.

3. **Determining Golden Rule Level of Capital using MPK and δ :** The slope of the production function is marginal product of capital (MPk) and the slope of depreciation is δ.
   (i) If MPk > δ, then the production function is steeper than the depreciation line. Therefore, increase in capital increases output more than depreciation, so consumption increases.
   (ii) If MPK < δ, then the production function is flatter than the depreciation line. Therefore, increase in capital increases output less than depreciation, so consumption decreases.
   (iii) If MPk = δ then the production function is parallel to the depreciation line. Therefore, increase in capital increases output which is equal to depreciation so the consumption does not change. Thus this is the golden rule level of capital.

Note **There is only one saving rate which produces golden rule level of capital k*gold. Any change in the saving rate would shift the investment curve s.f (k) and would move the economy to a steady state with a lower level of consumption**

Note **The economy does not automatically gravitate toward the Golden Rule steady state. If we want any particular steady state capital stock such as Golden Rule, we need a particular saving rate to support it.**

4. **Mathematical Derivation of Golden Rule Level of Capital :**

We know that,

$c = y - i$

$\Rightarrow \quad c = f(k) - s.f(k)$

At steady state level, k^*

i = Depreciation, *i.e.*, δk^*

$\Rightarrow \quad s.f(k^*) = \delta k^*$

$\therefore \quad c^* = f(k^*) - \delta k^*$

Differentiating both sides w.r.t k^*

$$\partial c^*/\partial k^* = \partial f(k^*)/\partial k^* - \delta$$
$$= \text{MPk} - \delta$$

First order condition, for maximum consumption is

$\partial c^*/\partial k^* = 0$

$\Rightarrow$ MPk – $\delta = 0$

$\Rightarrow$ MPk = δ

5. **Transition to the Golden Rule level of Capital :** We have assumed that the policy maker can simply choose the economy steady state and can easily go there so the policy maker would choose such saving rate where the level of consumption is highest, *i.e.*, golden rule steady state. If economy is already at the golden rule steady state level then there are no issues for the policy makers but if the economy has reached a steady state level other than the golden rule level then there are two possibilities.

(i) Starting with too much capital, *i.e.*, capital is more than the golden rule.

(ii) Starting with two little capital, *i.e.*, capital is less than the golden rule.

Case I : Starting with too much capital : If the economy has capital which is more than the Golden rule Steady than the policy maker should aim at reducing the saving rate such that capital is reduced and becomes equal to golden rule level. If the policy maker reduces the saving rate at time t_0 then there will be a sudden increase in consumption and decrease in investment. Because investment and depreciation were equal in the initial steady state, So depreciation will be more than investment and now the economy will not be in steady state. Gradually, It will cause a reduction in capital stock and consequently in output, consumption and investment. These variables will keep on falling until the economy reaches a new steady state level which we have assumed is the golden rule level so here the consumption should be more than the original consumption.

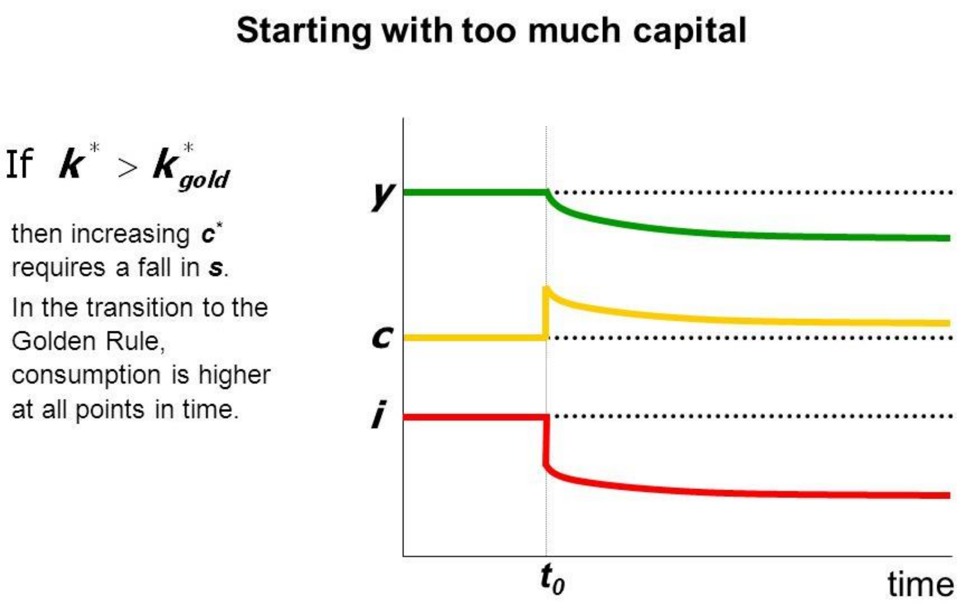


Figure 1 : Transition to Golden Rule Level of Capital

Note **Compared to the old steady state, consumption is higher not only in the new steady state but also along the entire path. So when capital is above the golden rule level, reduction is saving is a good economic policy.**

Case II : Starting with too little capital : If the economy has less capital then to reach the golden rule level of capital, the policymaker should increase the saving rate. When policy maker increases the saving rate the capital will start rising which will increase the output per worker. The consumption will fall initially but gradually it will rise because output is rising and finally consumption reaches a level which is more than the initial level of consumption. Suppose at time t_0 the policy maker increases the saving rate so the consumption will fall immediately. The increase in saving rate causes increase in capital stock over time which causes increase in output and hence increase in consumption. So, eventually economy reaches the golden rule level of capital where the consumption is maximum.

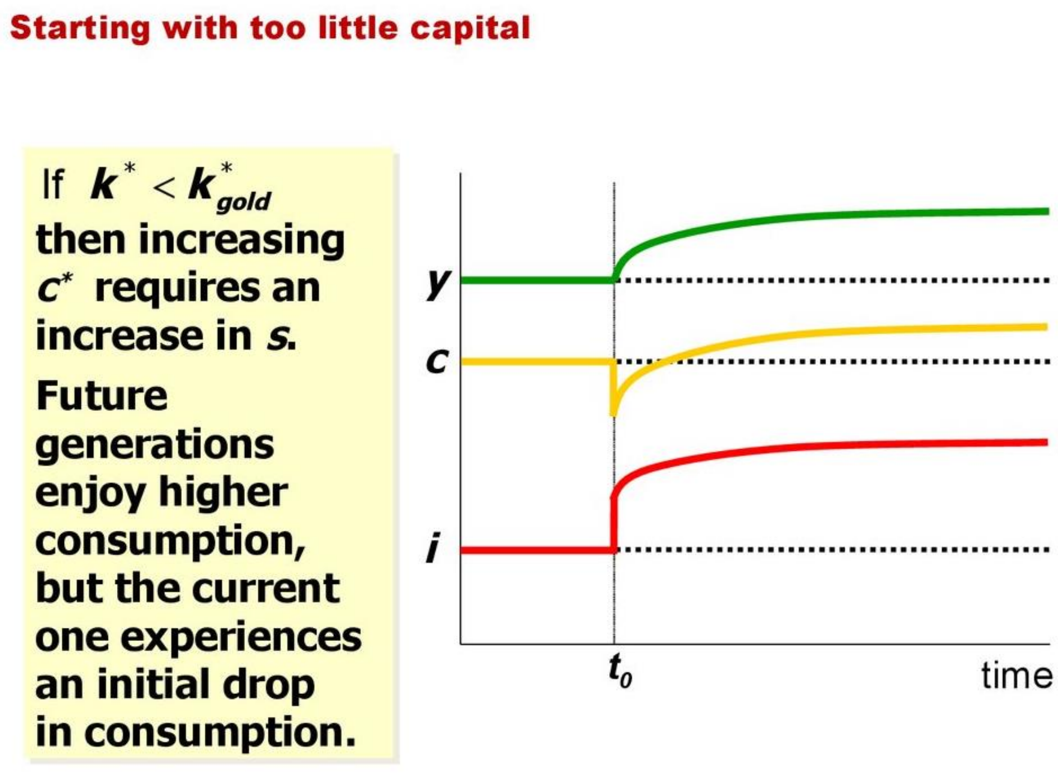


Figure 2 : Transition to Golden Rule Level of Capital

6. **Golden Rule Level of Capital with Population Growth :** Golden rule level of Capital is obtained where marginal productivity of capital is equal to the sum of depreciation rate and population growth rate, *i.e.*, MPk = $\delta + n$.

7. **Changing the Rate of Saving to Reach Golden Rule Level of capital :** If the level of capital is below the golden rule level of capital then the policy maker

should increase the saving rate. Higher saving means higher public saving or higher private saving or the combination of both.

Public Savings : The most direct way in which the government can influence the national savings is through Public savings. It is the difference between the revenue of the government and the expenditure of the government. When the expenditure of the government exceeds its revenue then we call it budget deficit. And it causes negative public savings because a budget deficit increases the interest rate and crowds out the private investment which reduces the capital stock whereas if the government revenue are more than the expenditure of the government then this budget surplus causes interest rates to fall and stimulate private investment which increases the capital stock.

Private Savings : The government can indirectly affect the national savings by influencing the private savings. Private saving are done by households or firms. How much the people will save depends upon the incentive they have to save. Government can provide the incentive to private saving by allowing the tax concession.

Exercise 3

Theory Questions

Q1. In the Solow model with no population growth and no technological progress, explain the concept of Golden Rule level of consumption. Use a diagram to discuss your answer. **[Eco. (H) 2011]**

Q2. Define the Golden rule level of capital $\left(K^*_{gold}\right)$.

Q3. What is meant by Golden Rule Steady State? How does the economy get there if it has smaller amount of initial capital?

Q4. Explain short run and long run implications of increasing capital per worker to reach golden level.

Q5. Suppose that in the Solow model without technological progress, an economy begins with less capital than in the Golden Rule Steady State. The saving rate at time t_0 rises to the golden Rule level. Show the time path of adjustment of output per worker, consumption per worker and investment per worker. **[Eco. (H) 2008]**

Q6. Suppose that the economy has reached a steady state with more capital than it would have in the golden rule steady state, what is the transition to the golden rule steady state.

Q7. What is the Golden Rule level of capital? Explain the policy interventions required to achieve the Golden Rule level if the actual capital stock differs from the Golden Rule level. **[Eco. (H) 2015]**

Q8. What do you understand by golden rule steady state ? Derive the condition for golden rule steady state. **[Eco. (H) 2017]**

Q9. Derive the Golden Rule Level of capital in the basic Solow model using diagram and equations. Explain the transition paths of output, consumption and investment per worker when saving rate is increased to achieve the Golden Rule level. **[Eco. (H) 2013]**

Q10. An economy is at a steady state with less capital than the golden rule steady state. Show the transition paths of output, consumption and investment towards the golden rule steady state. **[Eco. (H) 2017]**

Q11. Describe graphically how an economy attains golden rule of capital accumulation. If an economy starts with too little capital, show the transition path of output per worker, consumption per worker, and investment per worker to reach the golden rule steady state. **[Eco. (H) 2019]**

Numerical Problems

Q1. Consider an economy described by the following production function :

$$Y = f(K,\ L) = K^{1/2}L^{1/2}$$

(i) Assume that the rate of depreciation is 10% per year and rate of saving is 30%. Find the steady state capital stock per worker of Solow's model.

(ii) Does your answer to (i) also provide the golden rule steady state solution?

(iii) What is the steady state level of investment?

[**Ans. :** (i) $y = 3$, $k^* = 9$, (ii) $k^*_{gold} = 25$, (iii) $i = 0.9$]

Q2. Consider an economy described by the following production function :

$$Y = f(K,L) = K^{2/3}L^{1/3}$$

(i) Assume that the rate of depreciation is 10% per year and rate of saving is 30%. Find the steady state capital stock per worker of Solow's model.

(ii) Does your answer to (i) also provide the golden rule steady state solution?

(iii) What is the steady state level of investment? **[Eco. (H) 2009]**

[**Ans. :** (i) $y = 9$, $k^* = 27$, (ii) $k^*_{gold} = 296.296$, (iii) $i = 2.7$]

Q3. Consider an economy described by the following production function :

$$Y = f(K,\ L) = K^{1/4}L^{3/4}$$

(i) Assume that the rate of depreciation is 20% per year and rate of saving is 25%. Find the steady state capital stock per worker of Solow's model.

(ii) Does your answer to (i) also provide the golden rule steady state solution?

(iii) What is the steady state level of investment?

[**Ans. :** (i) $y = (1.25)^{1/3}$, $k^* = (1.25)^{4/3}$, (ii) $k^*_{gold} = (1.25)^{4/3}$,

(iii) $i = \dfrac{(1.25)^{4/3}}{5}$]

Q4. In the Solow model, without technical change but a constant rate of growth of population, assume an economy in steady state at time t_0. Let the economy attain the golden rule level of consumption per head at t_1. At t_0, let c_0, k_0 and y_0 be the levels of consumption, capital stock and output per head respectively.

(i) Derive and explain the condition which characterizes the golden rule.

(ii) If k_0 is greater than the capital stock per head at golden rule, show that the economy is dynamically inefficient by reducing the rate of savings. Consumption per head can be increased at all points of time : in the open time interval (t_0, t_1) and in the semi open time interval (t_1, ∞).

(iii) If k_0 is less than the capital stock per head at golden rule, increasing the rate of savings to reach the golden rule, involves an intergenerational tradeoff,

[Eco. (H) 2010]

Q5. An economy has the production function Y = 0.5 $\sqrt{K}\,\sqrt{L}$. **[Eco. (H) 2018]**

(i) What is the per worker production function?

(ii) In terms of the saving rate 's' and the depreciation rate 'δ' derive steady state levels of capital per worker, output per worker and consumption per worker.

(iii) Suppose that δ = 5%. What is the steady state output per worker and consumption per worker when s = 10%.

(iv) What is the Golden Rule steady state level capital stock per worker when δ = 5%?

[Ans : (i) y = 0.5 $\sqrt{k}$, (ii) $k^* = \frac{s^2}{4\delta^2}, y^* = \frac{s}{4\delta}, c^* = (1-s)\frac{s}{4\delta}$,

(iii) $k^* = 1$, $y^* = 1/2$, $c^* = 0.45$, (iv) $k^*_{gold} = 25$]

Q6. If the per capita production function is $y = k^{0.4}$, the savings rate is 0.5 and rate of depreciation is 0.1, what are the values of the capital and output per capita at steady state? If the economy intends to maximise its consumption per capita at steady state what should the level of savings be? With the help of a transition diagram trace the trajectory of the economy from the current state to the one where it maximises its consumption per capita at steady state. **[Eco. (H) 2021]**

Q7. In the context of the Solow model with no technical change, what is the saving rate that maximizes steady state consumption per worker? (Assume that consumption is equal to output minus investment). What is the marginal product of capital in this steady-state? **[Eco. (H) 2022]**

Basic Concepts

1. **Technological Progress in the Solow Model :** In the Solow Model, we considered production function as Y = F(K, L). To incorporate the technological process we have to modify the production function and we have to write the function as Y = F(K, L × E). Here, variable E is called ***efficiency of labour***. As the available technology improves the efficiency of labour also increases so effectively the output per worker increases. The term L X E measures the ***effective number of workers***. For eg. If the firm had 10 workers producing 100 units /day, *i.e.*, total output of 1000 unit/day. Now, with technological progress each worker produces 110 units/day, so total output with 10 worker becomes 1100 units/day. This is as good as having one additional worker/day. So technological process, *i.e.*, increase in the efficiency of labour is analogous to increase in labour force.

2. **Labour Augmenting :** The simplest assumption about technological progress is that it causes the efficiency of labour E to grow at some constant rate '*g*'. This form of technological progress with constant growth rate in efficiency or labour is called labour augmenting. The constant growth rate '*g*' in the efficiency of labour is called the rate of labour augmenting technological progress.

3. **The Steady State with Technological Progress :** Although technological progress does not cause the actual number of workers to increase, each worker in effect comes with more units of labour over time. Thus, technological progress causes the effective number of workers to increase.

When we assumed that technological progress is constant then we found economic variables in terms of per worker k = K/L & y = Y/L. Now since technological progress is not constant so we find the economic variables in terms of per effective worker, *i.e.*,

k = K/(L x E) represents capital per effective worker.

and y = Y/(L x E) represents output per effective worker.

So, now our production function becomes

Y = F[k, L x E] or $y = f(k)$

and $i = s.f(k)$ is investment per effective worker and the break even investment is $(\delta + n + g)k$, *i.e.*, the level of investment which is required to keep capital per effective worker constant.

The equation showing the evolution of k over time now changes to

$$\Delta k = s.f(k) - (\delta + n + g)k$$

This equation implies, that the change in the capital stock (Δk) equals investment $s.f(k)$ minus break even investment $(\delta + n + g)\,k$. In the steady state, investment $s.f(k)$ exactly offsets the reductions in k attributable to depreciation, population growth and technological progress. So, at steady level $\Delta k = 0$.

$\Rightarrow \quad s.f(k) = (\delta + n + g)k$

So, the inclusion of technological progress does not substantially alter our analysis of the steady state. there is one level of k, denoted k*, at which capital per effective worker and output per effective worker are constant.

4. **Sustained Growth :** Output per actual worker Y/L = $y \times$ E, because y is constant in the steady state and E is growing at rate g, so output per worker must also be growing at rate g in the steady state. That is, we have shown that technological progress can lead to sustained growth in output per worker. Once the economy is in the steady state, the rate of growth of output per worker depends only on the rate of technological progress. Thus, *according to the Solow model, only technological progress can explain sustained growth and persistently rising living standards*.

5. **Golden Rule Level of Capital with Technological Progress :** The introduction of technological progress also modifies the criterion for the Golden Rule. The golden rule level of capital is now defined as the steady state that maximizes consumption per effective worker. The steady state consumption per effective worker is

$$c^* = f(k^*) - (\delta + n + g)k^*$$

steady state consumption is maximized if

MPK = $\delta + n + g$, or MPK $- \delta = n + g$

that is at the Golden Rule level of capital, the net marginal product of capital, MPK $- \delta$, equals the rate of growth of output, $n + g$.

6. **Balanced Growth :** According to the Solow model, technological progress causes the values of many variables to rise together in the steady state, this property is

known as balanced growth. According to Solow model, in the steady state the output per worker and capital per worker both of these variables grow at *g*, which is the rate of technological progress. Technological progress also affects factor prices. In the steady state, the real wage grows at the rate of technological progress. However, the rental price of capital remains constant.

Exercise 4

Theory Questions

Q1. According to Solow, technological progress is the source of sustained per capita growth. Explain. **[Eco. (H) 2007]**

Q2. Explain how according to the Solow model, technological progress is the source of sustained increase in output per worker. **[Eco. (H) 2012, 2014]**

Q3. Show that in the steady state of Solow model with population growth and technological progress :

(i) The capital output ratio is constant;

(ii) Total capital income and total labour income grow at the rate of population growth plus the rate of technological progress, $n + g$. **[Eco. (H) 2008, 2014]**

Q4. What is the steady state growth rate of output per worker and that of total output when the technological progress is also taken into account?

Q5. When would it be possible to increase output per unit of labour even at the steady state? Discuss. **[Eco. (H) 2011]**

Q6. Explain how in the Solow model economy's level of output is affected by increase in population and technological progress.

Q7. Define Golden Rule Level of capital. Draw a diagram and derive the criterion for Golden Rule Level of capital when the economy has population growth and technological progress. Explain the need for savings rate to change for achieving the golden rule level of capital. **[Eco. (H) 2012]**

Numerical Problems

Q1. Let the production function of an economy be described by the following equation:

$$Y = F(K, AL) = K^a.(AL)^{1-a},$$

Where A represents labour augmenting (Harrod neutral) technological progress, K represents the capital stock and L the labour force, a is a constant less than 1. Assume that there are diminishing returns to both capital and effective labour (AL).

(i) Rewrite the equation in its intensive form, i.e., in terms of output and capital per efficiency unit of labour.

(ii) Define a steady state. Derive the condition which characterizes the steady state in this model.

(iii) What is the steady state growth rate of output per worker? How do the variables A, *s*, *n* and δ effect the level of output per worker? Do any of these variables have an impact on the rate of growth of output per capita? (symbols have their usual interpretations). **[Eco. (H) 2010]**

Basic Concepts

1. **Total Factor Productivity :** We have seen in the Solow model that sustained growth occurs only in the presence of technological progress. Without technological progress, capital accumulation runs into diminishing returns. With technological progress, however, improvements in technology continually offset the diminishing returns to capital accumulation. Labor productivity grows as a result, both directly because of the improvements in technology and indirectly because of the additional capital accumulation these improvements make possible.

Solow performed a simple accounting exercise to break down growth in output into growth in capital, growth in labor, and growth in technological change. This "growth accounting" exercise begins by postulating a production function such as

$$Y = BK^{\alpha}L^{1-\alpha}$$

where B is a Hicks-neutral productivity term. Taking logs and differentiating this production function, one derives the key formula of growth accounting:

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} + \frac{\dot{B}}{B}$$

This equation says that output growth is equal to a weighted average of capital and labor growth plus the growth rate of B. This last term, $\frac{\dot{B}}{B}$, is commonly referred to as total factor productivity growth or multifactor productivity growth.

Since we are primarily interested here in the growth rate of output per worker instead of total output, it is helpful to rewrite the above equation by subtracting L/L from both sides:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \frac{\dot{B}}{B}$$

That is, the growth rate of output per worker is decomposed into the contribution of physical capital per worker and the contribution from multifactor productivity growth.

Exercise 5

Total Factor Productivity

Q1. If the rate of growth of output in an economy is 7% and the rate of growth of capital and labour are 3% and 4% respectively for an economy with a production function $Y = AK^{0.6}L^{0.4}$, calculate the rate of growth of total factor productivity.

[Eco. (H) 2021]